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CATEGORICITY OF AN ABSTRACT ELEMENTARY CLASS IN TWO SUCCESSIVE CARDINALS

BY

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ABSTRACT

We investigate categoricity of abstract elementary classes without any remnants of compactness (like non-definability of well ordering, existence of E.M. models, or existence of large cardinals). We prove (assuming a weak version of GCH around λ) that if $\mathfrak{K}$ is categorical in λ, λ^+, $LS(\mathfrak{K}) \leq \lambda$ and has intermediate number of models in λ^{++}, then $\mathfrak{K}$ has a model in λ^{+++}.

ANNOTATED CONTENTS

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0. Introduction

Makowski [Mw85] is a readable and good exposition concerning categoricity in abstract elementary classes around $\aleph_1$.

Our primary concern is:

PROBLEM 0.1: *Can we have some (not necessarily much) classification theory for reasonable non-first order classes $\mathfrak{K}$ of models, with no uses of even traces of compactness and only mild set theoretic assumptions?*

Let me try to clarify the meaning of Problem 0.1.

What is the meaning of "mild set theoretic assumptions?" We are allowing requirements on cardinal arithmetic like GCH and weaker relatives. Preferably, assumptions like diamonds and squares and even mild large cardinals will not be used (apart from cases provable in ZFC, or in ZFC plus allowable assumptions).

In fact we try to continue [Sh 88], where results about the number of non-isomorphic models in $\aleph_1$ and $\aleph_2$ of a sentence $\psi \in L_{\omega_1,\omega}$ are obtained. Now in [Sh 88] the theorem parallel to the present one is proved assuming $2^{\aleph_0} < 2^{\aleph_1}$, so it is quite natural to use such assumptions here.

What is the meaning of "some classification theory?" While the dream is to have a classification theory as "full" as the one obtained in [Sh c], we will be glad to have theorems speaking just on having few models in some cardinals or even categoricity and at least one model in others. E.g. by [Sh 88] if $\psi \in L_{\omega_1,\omega}$ satisfies $1 \leq I(\aleph_1, \psi) < 2^{\aleph_1}$ (and $2^{\aleph_0} < 2^{\aleph_1}$) then $I(\aleph_2, \psi) > 0$. Here $I(\mu, \mathfrak{K})$ is the number of models in $\mathfrak{K}$ of cardinality μ, up to isomorphism.

What are "reasonable non-first order classes?" This means we allow classes of "locally finite" or "atomic" structures, or structures "omitting a type", or more generally the class of models of a sentence in $L_{\kappa,\omega}$, (i.e. allowing conjunction $< \kappa$ but quantification only over a finite string) but not one restricting ourselves to e.g. well orderings. In fact, we use "abstract elementary classes" from [Sh 88] (reviewed below).

What is the meaning of "uses traces of compactness?" For non-first order classes we cannot use the powerful compactness theorem, but there are many ways to get weak forms of it: one way is using large cardinals (compact cardinals in Makkai Shelah [MaSh 285], or just measurable cardinals as in Kolman Shelah [KlSh 362], and in [Sh 472]). Another way is to use "non-definability of well ordering" which follows from the existence of Ehrenfeucht–Mostowski models, and also from $\psi \in L_{\omega_1,\omega}$ having uncountable models (used extensively in [Sh 88]). Our aim is to use none of this and we would like to see if any theory is left.

Above all, we hope the proofs will initiate classification theory in this case, so we hope the flavour will be one of introducing and investigating notions of a model theoretic character. Proofs of, say, a descriptive set theory character, will not satisfy this hope.

It seems to us that this goal is met here. We prove (see 6.12):

THEOREM 0.2: ($2^{\lambda} < 2^{\lambda^+} < 2^{\lambda^{++}}$). *Let $\mathfrak{K}$ be an abstract elementary class. If $\mathfrak{K}$ is categorical in λ, λ^+ and λ^{++} <u>then</u> $I(\lambda^{+3}, \mathfrak{K}) > 0$.*

Here $\lambda^{+3} = \lambda^{+++}$ and in general $\aleph_\alpha^{+\beta}$ means $\aleph_{\alpha+\beta}$.

Of course, the categoricity in three successive cardinals is a strong assumption. Now note that in [Sh 88], the categoricity in $\aleph_0$ is gained "freely", so the gap is smaller than seems at first glance. Still it is better to have

THEOREM 0.3: *If $\mathfrak{K}$ is categorical in λ and λ^+ with $\lambda > \aleph_0$ and $1 \le I(\lambda^{++}, \mathfrak{K}) < 2^{\lambda^{++}}$, <u>then</u> $I(\lambda^{+3}, \mathfrak{K}) > 0$ provided that $\lambda \ge \beth_\omega$ or there is no normal ideal on λ^+ (a very weak assumption).*

Note however:

(α) A silly point: at exactly one point in the proof of 0.3 we assume $\lambda > \aleph_0$ (in the proof of 4.6). This is silly as our intent is to prove for general λ what we know for $\lambda = \aleph_0$ by [Sh 88]; however, there we assume $\mathfrak{K}$ is $\mathrm{PC}_{\aleph_1,\aleph_0}$, a reasonable assumption, but one which is not assumed here. We shall complete this in [Sh 603, §4], so we do not mention the assumption $\lambda > \aleph_0$ in theorems relying on 4.6. Also $\lambda = \aleph_0$ can serve instead of $\lambda \ge \beth_\omega$ using [Sh 88].

(β) More seriously, at some point we assume toward a contradiction that $K_{\lambda^{+3}} = \emptyset$ in order to prove the density of the set of minimal triples. This is fine for proving Theorem 0.2, but is not desirable if we want to develop a classification theory. This will be dealt with in [Sh 615].

(γ) Concerning $\mu_{\mathrm{wd}}(\lambda^{+2})$: in 6.12 we assume that $\aleph_0 < \lambda < \beth_\omega$ and there is no normal ideal on λ^+ (or just a specific normal ideal, as the one of the weak diamond is not). This is a very weak set theoretic assumption, see [Sh F368] for elimination.

We present below, as background, the following open questions which appeared in [Sh 88], for $\mathfrak{K}$ an abstract elementary class, of course, e.g. the class of models of $\psi \in L_{\kappa^+,\omega}$ with the relation $M \le_{\mathfrak{K}} N$ being $M \prec_{\mathcal{L}} N$ for $\mathcal{L}$ a fragment of $L_{\kappa^+,\omega}$ to which ψ belongs. In [Sh 87a], [Sh 87b], [Sh 88] we prove:

$(*)_3$ categoricity (of $\psi \in L_{\omega_1,\omega}(Q)$) in $\aleph_1$ implies the existence of a model of ψ of cardinality $\aleph_2$;

$(*)_4$ if $n > 0, 2^{\aleph_0} < 2^{\aleph_1} < \cdots < 2^{\aleph_n}, \psi \in L_{\omega_1,\omega}$ and $1 \le I(\aleph_\ell, \psi) < \mu_{wd}(\aleph_\ell)$ for $1 \le \ell \le n$, <u>then</u> ψ has a model of cardinality $\aleph_{n+1}$.

Now the problems were:

Problem (1) Prove $(*)_3, (*)_4$ in the context of an abstract elementary class $\mathfrak{K}$ which is $PC_{\aleph_0}$.
Problem (2) Parallel results in ZFC; e.g. prove $(*)_3$ when $2^{\aleph_0} = 2^{\aleph_1}$.
Problem (3) Construct examples; e.g. $\mathfrak{K}$ (or $\psi \in L_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \ldots, \aleph_n$ but not in $\aleph_{n+1}$.
Problem (4) If $\mathfrak{K}$ is PC_λ (and is an abstract elementary class†), and is categorical in λ and λ^+, does it necessarily have a model in λ^{++}?

Concerning Problem 3, by Hart Shelah [HaSh 323],2.10(2) + 3.8 there is $\psi_n \in \mathcal{L}_{\omega_1,\omega}$ categorical in $\aleph_0, \aleph_1, \ldots, \aleph_{k-1}$, but not categorical in λ if $2^\lambda > 2^{\aleph_{k-1}}$.

The direct motivation for the present work is that Grossberg asked me (Oct. 94) some questions in this neighborhood, in particular:
Problem (5) Assume $K =$ Mod(T) (i.e. K is the class of models of T), $T \subseteq L_{\omega_1,\omega}, |T| = \lambda$, $I(\lambda, K) = 1$ and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$. Does it follow that $I(\lambda^{++}, K) > 0$?

We think of this as a test problem and much prefer a model theoretic to a set theoretic solution. This is closely related to Problem 4 above and to [Sh 88, Theorem 3.7] (where we assume categoricity in λ^+, do not require $2^\lambda < 2^{\lambda^+}$ but take $\lambda = \aleph_0$ or some similar cases) and [Sh 88, Theorem 5.17(4)] (and see [Sh 88, 5.1,4.5] on the assumptions) (there we require $2^\lambda < 2^{\lambda^+}, 1 \leq I(\lambda^+, K) < 2^{\lambda^+}$ and $\lambda = \aleph_0$).

As said above, we are dealing with a closely related problem. Problem 0.1 was stated *a posteriori* but is, I think, the real problem.

In a first try we used more set theory, i.e. we used the definitional weak diamond on both λ^+ and λ^{++} (see Definition 2.13) and things like "a nice equivalence relation on $\mathcal{P}(\lambda)$ has either few or many classes" (see §2). Here we take a model theoretic approach.

We feel that this paper provides a reasonable positive solution to Problem 0.1, with a classification theory flavor. We shall continue in [Sh 600] toward a parallel of [Sh 87a], [Sh 87b]. Grossberg and Shelah, in the mid-eighties, started to write a paper (following [Sh 87a], [Sh 87b]) to prove that: if $\psi \in L_{\lambda^+,\omega}$ has models of arbitrarily large cardinality, and is categorical in μ^{+n} for each n and if $\mu \geq \lambda$ and $2^{\mu^{+n}} < 2^{\mu^{+n+1}}$ for $n < \omega$, then ψ is categorical in every $\mu' \geq \mu$; this is a weak form of the upward part of Los' conjecture. See Makkai Shelah [MaSh 285] on $T \subseteq \mathcal{L}_{\kappa,\omega}$ where κ is a compact cardinal; where we get downward and upward theorems for successor cardinals which are sufficiently bigger than $\kappa + |T|$. On the

† With $LS(\mathfrak{K}) \leq \lambda$ of course.

downward part, see [KlSh 362], [Sh 472] which deals with a downward theorem for successor cardinals which are sufficiently larger than $\kappa + |T|$ when the theory T is in the logic $\mathcal{L}_{\kappa,\omega}$ and κ is measurable. See also [Sh 394] which deals with abstract elementary classes with amalgamation, getting similar results with no large cardinals.

Part of §1 and §3 have a combinatorial character. Most of the paper forms the content of a course given in Fall '94 (essentially without §3, §9, §10). The paper is written with an eye to developing the model theory, rather than just proving Theorem 0.2.

0.4 CONJECTURE: *Any abstract elementary class with arbitrarily large models is categorical in every large enough cardinality or is not categorical (but has a model) in every large enough cardinality (probably in* ZFC*).*

* * *

On abstract elementary classes see [Sh 88] and [Sh 300, II, §3]. To make the paper self-contained, we will review some relevant definitions and results. We thank Gregory Cherlin for improving the writing of §1–§3 and for breaking 3.25, 3.26 and 3.27 into three proofs.

This work is continued in [Sh 600] and more.

Review: Abstract elementary classes

0.5 CONVENTIONS: $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$, where K is a class of τ-models for a fixed vocabulary $\tau = \tau_K = \tau_{\mathfrak{K}}$ and $\leq_{\mathfrak{K}}$ is a two-place relation on the models in K. We do not always strictly distinguish between K and $(K, \leq_{\mathfrak{K}})$. We shall assume that $K, \leq_{\mathfrak{K}}$ are fixed, and $M \leq_{\mathfrak{K}} N \Rightarrow M, N \in K$; and we assume that the following axioms hold. When we use $<_{\mathfrak{K}}$ in the sense of elementary submodel for first order logic, we write $<_{\mathcal{L}_{\omega,\omega}}$.

0.6 Definition: $\mathfrak{K}$ is called an abstract elementary class if:

Ax 0: The validity of $M \in K$ or of $N \leq_{\mathfrak{K}} M$ depends on N and M only up to isomorphism—in the second case, isomorphism of the pair.

Ax I: If $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

Ax II: $\leq_{\mathfrak{K}}$ is transitive and reflexive on K.

Ax III: If λ is a regular cardinal, M_i $(i < \lambda)$ is $\leq_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinals $\delta < \lambda$ we have $M_\delta = \bigcup_{i<\delta} M_i$) <u>then</u> $M_0 \leq_{\mathfrak{K}} \bigcup_{i<\lambda} M_i$.

Ax IV: If λ is a regular cardinal, $M_i\,(i<\lambda)$ is $\leq_{\mathfrak{K}}$-increasing continuous, $M_i \leq_{\mathfrak{K}} N$ then $\bigcup_{i<\lambda} M_i \leq_{\mathfrak{K}} N$.

Ax V: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 0,1$, then $M_0 \leq_{\mathfrak{K}} M_1$.

Ax VI: There is a cardinal λ such that: if $A \subseteq N$ and $|A| \leq \lambda$ then for some $M \leq_{\mathfrak{K}} N$ we have $A \subseteq |M|$ and $\|M\| \leq \lambda$. We define the Löwenheim–Skolem number $LS(\mathfrak{K})$ as the least such λ with $\lambda \geq |\tau|$. For simplicity we assume $M \in K \Rightarrow \|M\| \geq LS(\mathfrak{K})$.

Notation: $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu$.

$\mathcal{L}_{\omega,\omega}$ is first order logic.

A theory in $\mathcal{L}(\tau)$ is a set of sentences from $\mathcal{L}(\tau)$.

0.7 Definition: The embedding $f\colon N \to M$ is a $\mathfrak{K}$-embedding or a $\leq_{\mathfrak{K}}$-embedding if its range is the universe of a model $N' \leq_{\mathfrak{K}} M$ (so $f\colon N \to N'$ is an isomorphism (onto)).

Very central in [Sh 88], but peripheral here, is:

0.8 Definition: (1) For a logic $\mathcal{L}$ and vocabulary τ, $\mathcal{L}(\tau)$ is the set of $\mathcal{L}$-formulas in this vocabulary.

(2) Let T_1 be a theory in $\mathcal{L}_{\omega,\omega}(\tau_1), \tau \subseteq \tau_1$ vocabularies, Γ a set of types in $\mathcal{L}_{\omega,\omega}(\tau_1)$ (i.e. for some m, a set of formulas $\varphi(x_0,\ldots,x_{m-1}) \in \mathcal{L}_{\omega,\omega}(\tau_1)$). Then we let $EC(T_1,\Gamma) = \{M : M$ a τ_1-model of T_1 which omits every $p \in \Gamma\}$.

(3)
$$PC_\tau(T_1,\Gamma) = PC(T_1,\Gamma,\tau) = \{M : M \text{ is a } \tau\text{-reduct of some } M_1 \in EC(T_1,\Gamma)\}.$$

(4) We say that $\mathfrak{K}$ is PC^μ_λ if for some $T_1, T_2, \Gamma_1, \Gamma_2$ and τ_1 and τ_2 we have: T_ℓ is a first order theory in the vocabulary τ_ℓ, Γ_ℓ is a set of types in the vocabulary τ_ℓ and $K = PC(T_1,\Gamma_1,\tau_{\mathfrak{K}})$ and $\{(M,N) : M \leq_{\mathfrak{K}} N, M,N \in K\} = PC(T_2,\Gamma_2,\tau')$ where $\tau' = \tau_{\mathfrak{K}} \cup \{P\}$ (P a new one place predicate) and $|T_\ell| \leq |\tau_\ell| + \aleph_0 \leq \lambda, |\Gamma_\ell| \leq \mu$ for $\ell = 1,2$. If $\mu = \lambda$, we may omit it.

0.9 Example: If $\tau_1 = \tau, T_1, \Gamma$ are as above, and $(K, \leq_{\mathfrak{K}})$ is defined by $K =: EC(T_1,\Gamma)$, $\leq_{\mathfrak{K}} =: \prec_{\mathcal{L}_{\omega,\omega}}$, then the pair satisfies the Axioms from 0.6 and $LS(\mathfrak{K}) \leq |T_1| + \aleph_0 + |\tau_1|$.

0.10 Example: $V = L$. Let $\mathrm{cf}(\lambda) \geq \aleph_1, n < \omega$ then for some $\psi \in L_{\lambda^+,\omega}$ we have: ψ has no model of cardinality $\lambda^{+(n+1)}$, and is categorical in λ^{+n} (i.e. has one and only one model up to isomorphism).

Let $M^* = (L_{\lambda^{+n}}, \in, i)_{i \leq \lambda}$ and let ψ be

$$\bigwedge\{\varphi : \varphi \text{ is a first order sentence which } M^* \text{ satisfies}\}$$
$$\wedge (\forall x)\Big(x \in \lambda \equiv \bigvee_{i<\lambda} x = i\Big).$$

0.11 LEMMA: *Let I be a directed set (i.e. partially ordered by $\leq$, such that any two elements have a common upper bound).*
(1) If M_t is defined for $t \in I$, and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{K}} M_s$ then for every $t \in I$ we have $M_t \leq_{\mathfrak{K}} \bigcup_{s\in I} M_s \in \mathfrak{K}$.
(2) If in addition $t \in I$ implies $M_t \leq_{\mathfrak{K}} N$ then $\bigcup_{s\in I} M_s \leq_{\mathfrak{K}} N$.

Proof: By induction on $|I|$ (simultaneously for (1) and (2)), or see [Sh 88, 1.6]. $\blacksquare_{0.11}$

0.12 LEMMA:
(1) Let $\tau_1 = \tau \cup \{F_i^n : i < LS(\mathfrak{K})$ and $n < \omega\}, F_i^n$ an n-place function symbol (assuming, of course, $F_i^n \notin \tau$). If M_1 is an expansion of M to a τ_1-model and $\bar{a} \in {}^n|M|$ for some n, let $M_{\bar{a}}$ be the submodel of M with universe $\{F_i^n(\bar{a}) : i < LS(\mathfrak{K})\}$. Every model M from K can be expanded to a τ_1-model M_1 such that:
- *(A) $M_{\bar{a}} \leq_{\mathfrak{K}} M$ for any $\bar{a} \in {}^n|M|$ and $\bar{a} \in {}^n(M_{\bar{a}})$;*
- *(B) if $\bar{a} \in {}^n|M|$ then $\|M_{\bar{a}}\| \leq LS(\mathfrak{K})$;*
- *(C) if $\bar{b}$ is a subsequence of $\bar{a}$ (even up to rearrangement), then $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$;*
- *(D) for every $N_1 \subseteq M_1$ we have $N_1 \restriction \tau \leq_{\mathfrak{K}} M$ (this follows).*

(2) If $N \leq_{\mathfrak{K}} M$ is given, then we can choose the expansion M_1 so that clauses (A)–(D) hold and
- *(E) $N = N_1 \restriction \tau$ for some $N_1 \subseteq M_1$.*

Proof: We define, by induction on n, the values of $F_i^n(\bar{a})$ for every $i < LS(\mathfrak{K})$, $\bar{a} \in {}^n|M|$ such that $\bar{a} \subseteq N \Rightarrow M_{\bar{a}} \subseteq N$ when we are proving (2). By $Ax\,VI$ there is an $M_{\bar{a}} \leq_{\mathfrak{K}} M, \|M_{\bar{a}}\| \leq LS(\mathfrak{K})$ such that $|M_{\bar{a}}|$ includes

$$\bigcup\{M_{\bar{b}} : \bar{b} \text{ a subsequence of } \bar{a} \text{ of length } < n\} \cup \bar{a}$$

and $M_{\bar{a}}$ does not depend on the order of $\bar{a}$. Let $|M_{\bar{a}}| = \{c_i : i < i_0 \leq LS(\mathfrak{K})\}$, and define $F_i^n(\bar{a}) = c_i$ for $i < i_0$ and $F_i^n(\bar{a}) = c_0$ for $i_0 \leq i < LS(\mathfrak{K})$ (so we can demand "F_i^n is symmetric").

Clearly our conditions are satisfied: if $\bar{b}$ is a subsequence of $\bar{a}$ then $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}}$ by $Ax\,V$. Clearly clause (D) (hence clause (E)) holds by 1.11(2). $\blacksquare_{0.12}$

0.13 Lemma: *(1) There is a set Γ of types in $\mathcal{L}_{\omega,\omega}(\tau_1)$ (where τ_1 is as in Lemma 0.12) such that $K = PC(\emptyset, \Gamma, \tau)$. So K is a $PC^{2^{LS(\mathfrak{K})}}_{LS(\mathfrak{K})}$-class, see Definition* 0.8(4). *The types above consist of quantifier-free formulas and even basic ones.*
(2) Moreover, if $N_1 \subseteq M_1 \in EC(\emptyset, \Gamma)$, and N, M are the τ-reducts of N_1, M_1 respectively <u>then</u> $N \leq_{\mathfrak{K}} M$. Also, if $N \leq_{\mathfrak{K}} M$ <u>then</u> there is an τ_1-expansion M_1 of M and a submodel N_1 of M_1 such that $M_1 \in EC(\emptyset, \Gamma)$ and $N_1 \restriction \tau = N$.
(3) Also $\{(M, N) : N \leq_{\mathfrak{K}} M\}$ is a $PC^{(2^{LS(\mathfrak{K})})}_{LS(\mathfrak{K})}$-class, hence $\mathfrak{K}$ is as well.

Proof: (1) Let Γ_n be the set of complete quantifier free n-types $p(x_0, \ldots, x_{n-1})$ in $\mathcal{L}_{\omega,\omega}(\tau_1)$ such that: if M_1 is an L_1-model, $\bar{a}$ realizes p in M_1 and M is the L-reduct of M_1, then $M_{\bar{b}} \leq_{\mathfrak{K}} M_{\bar{a}} \in \mathfrak{K}$ for any subsequence of $\bar{b}$ of $\bar{a}$. Recall that $M_{\bar{c}}$ (for $\bar{c} \in {}^m|M_1|$) is the submodel of M whose universe is $\{F^m_i(\bar{c}) : i < LS(\mathfrak{K})\}$ (and there are such submodels) and subsequence include permutation.

Let Γ be the set of p which are complete quantifier free n-types for some $n < \omega$ in $\mathcal{L}_{\omega,\omega}(\tau_1)$ and which do not belong to Γ_n. So if M^1 is in $PC(\emptyset, \Gamma, \tau_1)$ then by 0.12 we have $M_1 \restriction \tau \in K$ hence $PC(\emptyset, \Gamma, L) \subseteq K$ and by 0.12(1) we have $K \subseteq PC(\emptyset, \Gamma, L)$.
(2) The first phrase is proven as in (1). For the second phrase, use 0.12(2).
(3) Follows from (2). $\blacksquare_{0.13}$

0.14 Conclusion: There is τ_1 with $\tau \subseteq \tau_1$ and $|\tau_1| \leq LS(\mathfrak{K})$ such that: for any $M \in K$ and any τ_1-expansion M_1 of M which is in $EC(\emptyset, \Gamma)$,

$$N_1 \subseteq M_1 \Rightarrow N_1 \restriction \tau \leq_{\mathfrak{K}} M,$$
$$N_1 \subseteq N_2 \subseteq M_1 \Rightarrow N_1 \restriction \tau \leq_{\mathfrak{K}} N_2 \restriction \tau.$$

0.15 Conclusion: If, for every $\alpha < (2^{LS(\mathfrak{K})})^+, K$ has a model of cardinality $\geq \beth_\alpha$ <u>then</u> K has a model in every cardinality $\geq LS(\mathfrak{K})$.

Proof: Use 0.13 and the value of the Hanf number for: models of a first order theory omitting a given set of types, for languages of cardinality $LS(K)$ (see [Sh c, VII, §5] and history there). $\blacksquare_{0.15}$

0.16 Definition: For λ regular $> LS(\mathfrak{K})$ and $N \in \mathfrak{K}_\lambda$ we say $\bar{N} = \langle N_\alpha : \alpha < \lambda \rangle$ is a representation of N if $\bar{N}$ is $\leq_{\mathfrak{K}}$-increasing continuous, $\|N_\alpha\| < \lambda$ and $N = \bigcup_{\alpha<\lambda} N_\alpha$. If $\lambda = \mu^+$ then, if not said otherwise, we require $\|N_\alpha\| = \mu$.

How will we define types and, in particular, the set $\mathcal{S}(M)$ of complete types over M, when no formulas are present? If we have a "monster model" $\mathfrak{C}$ we can use automorphisms; but any such "monster" is far down the road. So we will "chase diagrams" in K_λ (being careful not to use excessively large models). This gives us a relation of "having the same type" we call E^{at}_μ, *but* this relation in general is not transitive (if we do not have amalgamation in K_λ). So E_μ will be defined as the transitive closure of E^{at}_μ.

0.17 Definition: (1) The two-place relation E^{at}_M is defined on triples (M, N, a) with M fixed, $M \leq_{\mathfrak{K}} N \in K_{\|M\|}$, and $a \in N$ by:

$$(M, N_1, a_1) E^{at}_M (M, N_2, a_2) \text{ \underline{iff} there is } N \in K_{\|M\|} \text{ and } \leq_{\mathfrak{K}}\text{-embeddings } f_\ell : N_\ell \to N \text{ for } \ell = 1,2 \text{ such that: } f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).$$

Let E_M be the transitive closure of E^{at}_M.
(2) For $\mu \geq LS(\mathfrak{K})$ and $M \in K_\mu$ we define $\mathcal{S}(M)$ as $\{\mathrm{tp}(a, M, N) : M \leq_{\mathfrak{K}} N \in K_\mu \text{ and } a \in N\}$ where $\mathrm{tp}(a, M, N) = (M, N, a)/E_M$.
(3) We say "a **realizes** p in N" if $a \in N, p \in \mathcal{S}(M)$ and for some $N' \in K_\mu$ we have $M \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N, a \in N'$, and $p = \mathrm{tp}(a, M, N')$.
(4) We say "a_2 **strongly realizes** $(M, N^1, a^1)/E^{at}_M$ in N" if for some N^2, a^2 we have $M \leq_{\mathfrak{K}} N^2 \leq_{\mathfrak{K}} N, a_2 \in N^2$, and $(M, N^1, a^1)\, E^{at}_M\, (M, N^2, a^2)$.
(5) We say $M_0 \in \mathfrak{K}$ is an **amalgamation base** if letting $\lambda = \|M_0\|$ we have: for every $M_1, M_2 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathfrak{K}_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.
(6) We say $\mathfrak{K}$ is **stable** in λ <u>if</u> $LS(\mathfrak{K}) \leq \lambda$ and for all $M \in K_\lambda$ we have $|\mathcal{S}(M)| \leq \lambda$.

0.18 Observation: If M is an amalgamation base then $E_M = E^{at}_M$, and we have:

"a strongly realizes $(M, N, b)/E^{at}_M$ in N" iff "a realizes $(M, N, b)/E_M$".

0.19 Definition: (1) $\mathfrak{K}$ has the **λ-amalgamation property** or has amalgamation in λ, if every $M_0 \in \mathfrak{K}_\lambda$ is an amalgamation base (see 0.17(5) above).
(2) N is **universal** over M when: $M \leq_{\mathfrak{K}} N$ and if $M \leq_{\mathfrak{K}} N' \in K_{\leq \|N\|}$ then N' can be $\leq_{\mathfrak{K}}$-embedded into N over M; so M is an amalgamation base if $\|N\| = \|M\|$.
(3) $\mathfrak{K}$ has universal extensions in λ <u>if</u> for every $M \in K_\lambda$ there is N such that:
(a) $M \leq_{\mathfrak{K}} N \in K_\lambda$,
(b) N is universal over M.
(4) N_1, N_2 have a joint embedding if for some $N \in K$ there are $\leq_{\mathfrak{K}}$-embeddings h_ℓ of N_ℓ into N for $\ell = 1, 2$. Let JEP_μ (JEP) means this holds for N_1, N_2 in K_μ (in K).

(5) $\mathfrak{K}_\lambda$ has **unique (disjoint) amalgamation** (or $\mathfrak{K}$ has unique (disjoint) amalgamation in λ) when: if $M^\ell_i \in K_\lambda$ for $\ell < 2, i < 4$, and for $i = 1,2$ we have $h^\ell_{i,0}$ is a $\leq_{\mathfrak{K}}$-embedding of M^ℓ_0 into $M^\ell_i, h^\ell_{3,i}$ is a $\leq_{\mathfrak{K}}$-embedding of M^ℓ_i into $M^\ell_3, h^\ell_{3,1} \circ h^\ell_{1,0} = h^\ell_{3,2} \circ h^\ell_{2,0}$ and $\mathrm{Rang}(h^\ell_{3,1}) \cap \mathrm{Rang}(h^\ell_{3,2}) = \mathrm{Rang}(h^\ell_{3,1} \circ h^\ell_{1,0})$ and for $i < 3, f_i$ is an isomorphism from M^0_i onto $M^1_i, f_0 \subseteq f_1, f_0 \subseteq f_2$ then for some $N \in K_\lambda$ there are $\mathfrak{K}$-embedding $h_\ell : M^\ell_3 \to N$ such that $h_0 \circ h^0_{3,i} = h_1 \circ h^1_{3,i}$, for $i = 1,2$.

(6) Let $p_\ell \in \mathcal{S}(M_\ell)$ for $\ell = 0,1$. We say $p_0 \leq p_1$ if $M_0 \leq_{\mathfrak{K}} M_1$ and for some N and a we have $M_1 \leq_{\mathfrak{K}} N \in K_{\|M_1\|+LS(\mathfrak{K})}, a \in N$ and $\mathrm{tp}(a, M_\ell, N) = p_\ell$ for $\ell = 0,1$. We also write $p_0 = p_1 \restriction M_0$ (p_0 is unique knowing M_0, M_1, p_1 hence $p_1 \restriction M_0$ is well defined).

0.20 Claim:

(1) *If $\mathfrak{K}$ is categorical in λ (see Definition 0.25(2)) and $LS(\mathfrak{K}) \leq \lambda$ then: there is a model in K_{λ^+} iff for some (equivalently, every) model $M \in K_\lambda$ there is N such that $M <_{\mathfrak{K}} N \in K_\lambda$ iff for some (equivalently every) $M \in K_\lambda$ there is N such that $M <_{\mathfrak{K}} N \in K_{\lambda^+}$.*

(2) *If $\mathfrak{K}$ has amalgamation in $\lambda, LS(\mathfrak{K}) \leq \lambda$, and $M_0 \leq_{\mathfrak{K}} M_1$ are in K_λ with $M_0 \leq_{\mathfrak{K}} N_0 \in K_{\lambda^+}$ then we can find h and N_1 such that $N_0 \leq_{\mathfrak{K}} N_1 \in K_{\lambda^+}$ and h is a $\leq_{\mathfrak{K}}$-embedding of M_1 into N_1 extending id_{M_0}. We can allow $N_0 \in K_\mu$ with $\mu > \lambda$ if $\mathfrak{K}$ has amalgamation in every $\lambda' \in [\lambda, \mu)$.*

(3) *Assume $\mathfrak{K}$ has amalgamation in λ and $LS(\mathfrak{K}) \leq \lambda$. If $M_0 \leq M_1$ are from K_λ and $p_0 \in \mathcal{S}(M_0)$ then we can find an extension $p_1 \in \mathcal{S}(M_1)$ of p_0.*

Proof: (1) For "if", we can choose by induction on $i < \lambda^+$ models $M_i \in K_\lambda, \leq_{\mathfrak{K}}$-increasing continuous, $M_i \neq M_{i+1}$; for $i = 0$ use $K_{\lambda^+} \neq \emptyset$, for i limit take union, for $i = j + 1$ use the previous sentence; so $M_{\lambda^+} = \bigcup\{M_i : i < \lambda^+\} \in K_{\lambda^+}$ as required. For the "only if" direction use 0.12.

(2), (3) Left to the reader. $\blacksquare_{0.20}$

0.21 Remark: We can here add the content of 6.5, 6.7, 6.12.

0.22 Definition: (1) For $\lambda > LS(\mathfrak{K})$ we say "$N \in \mathfrak{K}$ is λ-**saturated**" if for every $M \leq_{\mathfrak{K}} N$ of cardinality $< \lambda$, if $M \leq_{\mathfrak{K}} N' \in K_{<\lambda}$ and $a' \in N'$ then some $a \in N$ strongly realizes $(M, N', a')/E^{at}_M$ (in the interesting cases $/E_M$ suffices).

(2) We say "$N \in \mathfrak{K}$ is λ-**saturated above** μ (or is λ-saturated $\geq \mu$)" if above we restrict ourselves to M of cardinality $\geq \mu$. If we omit λ we mean $\lambda = \|N\|$. (Cf. (λ, κ)-saturated in Definition 0.28, 0.29 below.)

0.23 FACT: (1) If $LS(\mathfrak{K}) \leq \mu < \lambda, \lambda$ is regular, $\mathfrak{K}$ has the amalgamation property in every $\mu' \in [\mu, \lambda)$, and for all $M \in K_{[\mu,\lambda)}$ we have $|\mathcal{S}(M)| \leq \lambda$ and $\lambda = \mathrm{cf}(\lambda)$, then there is some $M \in K_\lambda$ saturated above μ.
(2) Assume $\lambda > \mu \geq LS(\mathfrak{K})$ and $N \in K_{\geq\lambda}$ and $\mathfrak{K}$ has amalgamation in K_{μ_1}, for every $\mu_1 \in [\mu, \lambda)$. Then: N is λ-saturated above μ iff for every $M \leq_{\mathfrak{K}} N$ of cardinality $< \lambda$ but $\geq \mu$, every $p \in \mathcal{S}(M)$ is realized in N (i.e. for some $a \in N$ we have $\mathrm{tp}(a, M, N) = p$).
(3) If $LS(\mathfrak{K}) \leq \mu_0 \leq \mu_0' \leq \mu' \leq \mu$ and M is μ-saturated above μ_0, then it is μ'-saturated above μ_0'. If $LS(\mathfrak{K}) \leq \mu_0 < \mu$ then: M is μ-saturated above μ_0 iff for every $\lambda \in [\mu_0, \mu), M$ is λ^+-saturated above λ.

Proof: Check.

0.24 Definition: The type $p \in \mathcal{S}(M)$ is **local** when: for any directed partial order I and models $M_t \leq_{\mathfrak{K}} M$ for $t \in I$ with $I \models t \leq s \Rightarrow M_t \leq_{\mathfrak{K}} M_s$ and $M = \bigcup_{t \in I} M_t$, and any $p' \in \mathcal{S}(M)$ if ($p \restriction M_t = p' \restriction M_t$) for all $t \in I$ then $p = p'$. We say M is **local** if every $p \in \mathcal{S}(M)$ is, and $\mathfrak{K}$ is **local** if every $M \in \mathfrak{K}$ is. We can add "above μ" as in Definition 0.22(2).

0.25 Definition: (1) $I(\lambda, K) = I(\lambda, \mathfrak{K})$ is the number of $M \in K_\lambda$ up to isomorphism.
(2) $\mathfrak{K}$ (or K) is **categorical** in λ if $I(\lambda, K) = 1$.
(3) $IE(\lambda, \mathfrak{K}) = \sup\{|K'| : K' \subseteq K_\lambda$ and for $M \neq N$ in K'_λ, M is not $\leq_{\mathfrak{K}}$-embeddable into $N\}$. Abusing notation, if we write $IE(\lambda, K) \geq \mu$, we mean that for some $K' \subseteq K_\lambda$ as above, $|K'| \geq \mu$, and similarly for $= \mu$. If there is a problem with attainment of the supremum we shall say explicitly.

0.26 THE MODEL-HOMOGENEITY = SATURATIVITY LEMMA: *Let $\lambda > \mu \geq LS(\mathfrak{K})$ and $M \in K$.*
(1) M is λ-saturated above μ iff M is $(\mathcal{D}_{\mathfrak{K}}, \lambda)$-homogeneous above μ, which means: for every $N_1 \leq_{\mathfrak{K}} N_2 \in K$ such that $\mu \leq \|N_1\| \leq \|N_2\| < \lambda$ and $N_1 \leq_{\mathfrak{K}} M$, there is a $\leq_{\mathfrak{K}}$-embedding f of N_2 into M over N_1.
(2) If $M_1, M_2 \in K_\lambda$ are λ-saturated above $\mu < \lambda$ and for some $N_1 \leq_{\mathfrak{K}} M_1, N_2 \leq_{\mathfrak{K}} M_2$, both of cardinality $\in [\mu, \lambda)$, we have $N_1 \cong N_2$ then $M_1 \cong M_2$; in fact, any isomorphism f from N_1 onto N_2 can be extended to an isomorphism from M_1 onto M_2.
(3) If in (2) we demand only "M_2 is λ-saturated" and $M_1 \in K_{\leq\lambda}$ then f can be extended to a $\leq_{\mathfrak{K}}$-embedding from M_1 into M_2.
(4) In part (2) instead of $N_1 \cong N_2$ it suffices to assume that N_1 and N_2 can be $\leq_{\mathfrak{K}}$-embedded into some $N \in K$, which holds if $\mathfrak{K}$ has the JEP or just JEP$_\mu$.

Proof: (1) The "if" direction is easy as $\lambda > LS(\mathfrak{K})$. Let us prove the other direction.

By 0.23(3) without loss of generality λ is regular, moreover N_2 has cardinality $\|N_1\|$.

Let $|N_2| = \{a_i : i < \kappa\}$, and we know $\kappa =: \|N_1\| = \|N_2\| < \lambda$. We define by induction on $i \leq \kappa, N_1^i, N_2^i, f_i$ such that:

(a) $N_1^i \leq_{\mathfrak{K}} N_2^i$ and $\|N_2^i\| = \kappa$,

(b) N_1^i is $\leq_{\mathfrak{K}}$-increasing continuous in i,

(c) N_2^i is $\leq_{\mathfrak{K}}$-increasing continuous in i,

(d) f_i is a $\leq_{\mathfrak{K}}$-embedding of N_1^i into M,

(e) f_i is increasing continuous in i,

(f) $a_i \in N_1^{i+1}$,

(g) $N_1^0 = N_1, N_2^0 = N_2, f_0 = id_{N_1}$,

(h) N_1^i and N_2^i has cardinality κ.

For $i = 0$, clause (g) gives the definition. For i limit let:

$N_1^i = \bigcup_{j<i} N_1^j$ and

$N_2^i = \bigcup_{j<i} N_2^j$ and

$f_i = \bigcup_{j<i} f_j$.

Now (a)–(f) continues to hold by continuity (and $\|N_2^i\| < \lambda$ as λ is regular).

For i successor we use our assumption; more elaborately, let $M_1^{i-1} \leq_{\mathfrak{K}} M$ be $f_{i-1}(N_1^{i-1})$ and M_2^{i-1}, g_{i-1} be such that g_{i-1} is an isomorphism from N_2^{i-1} onto M_2^{i-1} extending f_{i-1}, so $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i-1}$ (but without loss of generality $M_2^{i-1} \cap M = M_1^{i-1}$). Now apply the saturation assumption with M, M_1^{i-1}, $\mathrm{tp}(g_{i-1}(a_{i-1}), M_1^{i-1}, M_2^{i-1})$ here standing for N, M, p there (note: $a_{i-1} \in N_2 = N_2^0 \subseteq N_2^{i-1}$ and $\lambda > \kappa = \|N_2^{i-1}\| = \|M_2^{i-1}\|$ and $\|M_1^{i-1}\| = \|N_1^{i-1}\| \geq \|N_1^0\| = \|N_1\| \geq \mu$ so the requirements including the requirements on the cardinalities in Definition 0.22 hold). So there is $b \in M$ such that $\mathrm{tp}(b, M_1^{i-1}, M) = \mathrm{tp}(g_{i-1}(a_{i-1}), M_1^{i-1}, M_2^{i-1})$. Moreover (if $\mathfrak{K}$ has amalgamation in μ the proof is slightly shorter) remembering the second sentence in 0.22(1) which speaks about "strongly realizes" there is $b \in M$ such that b strongly realizes $(M_1^{i-1}, M_2^{i-1}, g_{i-1}(a_{i-1}))/E^{at}_{M_1^{i-1}}$ in M. This means (see Definition 0.17(4)) that for some $M_1^{i,*}$ we have $M_1^{i-1} \leq_{\mathfrak{K}} M_1^{i,*} \leq_{\mathfrak{K}} M$ and $(M_1^{i-1}, M_2^{i-1}, g_{i-1}(a_{i-1})) E^{at}_{M_1^{i-1}} (M_1^{i-1}, M_1^{i,*}, b)$. This means (see Definition 0.17(1)) that $M_1^{i,*}$ too has cardinality κ and there is $M_i^{2,*} \in K_\kappa$ such that $M_1^{i-1} \leq_{\mathfrak{K}} M_2^{i,*}$ and there are $\leq_{\mathfrak{K}}$-embeddings h_2^i, h_1^i of $M_2^{i-1}, M_1^{i,*}$ into $M_2^{i,*}$ over M_1^{i-1} respectively, such that $h_2^i(g_{i-1}(a_{i-1})) = h_1^i(b)$.

Now changing names, without loss of generality h_2^i is the identity.

Let N_2^i, h_i be such that $N_2^{i-1} \leq_{\mathfrak{K}} N_2^i, h_i$ an isomorphism from N_2^i onto $M_2^{i,*}$ extending g_{i-1}. Let $N_1^i = h_i^{-1} \circ h_1^i(M_1^{i,*})$ and $f_i = (h_1^i)^{-1} \circ (h_i \restriction N_1^i)$.

We have carried the induction. Now f_κ is a $\leq_{\mathfrak{K}}$- embedding of N_1^κ into M over N_1, but $|N_2| = \{a_i : i < \kappa\} \subseteq N_1^\kappa$, so $f_\kappa \restriction N_2 : N_2 \to M$ is as required.

(2), (3) By the hence and forth argument (or see [Sh 300, II, §3] = [Sh h, II, §3]).

(4) Easy, too. $\blacksquare_{0.23}$

0.27 Remark: Note that by 0.26(2), if M is μ-saturated above μ_0 and $\mathfrak{K}$ has the JEP_{μ_0} then $\mathfrak{K}$ has λ-amalgamation for each $\lambda \in [\mu_0, \mu)$.

0.28 Definition: Fix $\lambda \geq \kappa$ with κ regular.

(1) We say that $N_1 \in K_\lambda$ is (λ, κ)-saturated over N_0 or that $(N_1, c)_{c \in N_0}$ is (λ, κ)-saturated (and λ-saturated of cofinality κ means (λ, κ)-saturated) if: there is a sequence $\langle M_i : i < \kappa \rangle$ which is $\leq_{\mathfrak{K}}$-increasing continuous with $M_0 = N, M_\kappa = N_1$ and $M_{i+1} \in K_\lambda$ universal over M_i (see 0.19(2), hence each M_i is an amalgamation base).

(2) If we omit κ, we mean $\kappa = \mathrm{cf}(\lambda)$; (λ, α)-saturated means $(\lambda, \mathrm{cf}(\alpha))$-saturated; and N_1 is $(\lambda, 1)$-saturated over N_0 means just $N_0 \leq_{\mathfrak{K}} N_1$ are in K_λ; "N_1 is (λ, κ)-saturated" means "for some $N_0 \in K_\lambda$, N_1 is (λ, κ)-saturated over N_0.

0.29 CLAIM: *Fix $\lambda \geq \kappa$ with κ regular.*

(1) *If N_1, N_2 are (λ, κ)-saturated over N, <u>then</u> N_1, N_2 are isomorphic over N (i.e. the (λ, κ)-saturated model over $N \in K_\lambda$ is unique over N).*

(2) *If $K_\lambda \neq \emptyset, \lambda \geq \kappa = \mathrm{cf}(\kappa)$ and over every $M \in K_\lambda$ there is N with $M \leq_{\mathfrak{K}} N \in K_\lambda$ universal over M, <u>then</u> for every $N \in K_\lambda$ there is $N_1 \in K_\lambda$ which is (λ, κ)-saturated over N.*

(3) *If $\mathfrak{K}_\lambda$ has amalgamation and $\mathfrak{K}$ is stable in λ (i.e. $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$) <u>then</u> every $M \in K_\lambda$ has a universal extension (so part (2)'s conclusion holds).*

Proof: See [Sh 300, Ch. II] or check ((3) is 0.32(3)(a)). $\blacksquare_{0.29}$

We do not need at present but recall from [Sh 88]:

0.30 CLAIM: *There is $\tau' \supseteq \tau \cup \{P_0, P_1, P_2, c\}$ of cardinality $\leq LS(\mathfrak{K})$ with c an individual constant, with P_ℓ unary predicates, and a set Γ of quantifier free types such that:*

(a) *if $M' \in PC_{\tau'}(\emptyset, \Gamma)$ and $M_\ell = (M' \restriction \tau) \restriction P_\ell^{M'}$ for $\ell = 0, 1, 2$, <u>then</u> $M_\ell \in K, M_0 \leq_{\mathfrak{K}} M_1, M_0 \leq_{\mathfrak{K}} M_2, c^{M'} \in M_2$, and $N' \subseteq M' \Rightarrow (N' \restriction \tau) \restriction P_\ell^{N'} \leq_{\mathfrak{K}} M_\ell$, and there is no $b \in M_1$ satisfying:*
   - $\otimes$ *for every $\bar{a} \in {}^{\omega>}(P_0^{M'})$, letting $N_{\bar{a}}$ be the τ'-submodel of M' generated by $\bar{a}$ and $M_{\bar{a}}^\ell = M_\ell \restriction (M_\ell \cap N_{\bar{a}})$, we have $M_{\bar{a}}^\ell \leq_{\mathfrak{K}} M_\ell$, and b strongly realizes $tp(c^{M'}, M_{\bar{a}}^0, M_{\bar{a}}^2)$ in $M_{\bar{a}}^1$,*

(b) *if* M_ℓ^* *for* $\ell = 0, 1, 2$ *and* c *are as in* (a) *and* $M_0 = M_1 \cap M_2$, *then for some* M' *we have clause* (a).

Proof: Should be clear; see [Sh 88], [Sh 394]. $\blacksquare_{0.30}$

0.31 Remark:
(1) Claim 1.26 enables us to translate results of the form: the existence of a two cardinal with omitting types model in λ_2 implies the existence of one in λ_1, provided that types are local in the sense that $p \in \mathcal{S}(M)$ is determined by $\langle p \restriction N : N \leq_{\mathfrak{K}} M, \|N\| \leq \lambda \rangle$.
(2) This enables us to prove implications between cases of λ-categoricity, if we have a nice enough theory of types as in [Sh c, VIII, §4]; if we have in λ_2 a saturated model, categoricity in λ_1 implies categoricity in λ_2. Also (if we know a little more) categoricity in λ_2 is equivalent to the non-existence of a non-saturated model in λ_2.

0.32 Claim: (1) *Assume* $M_n \leq_{\mathfrak{K}} M_{n+1}, M_n \in K_\lambda, \mathfrak{K}$ *has amalgamation in* λ. *If* $p_n \in \mathcal{S}(\mathcal{M}_n), p_n \leq p_{n+1}$ (*i.e.* $p_n = p_{n+1} \restriction M_n$, *see* 0.19(6)), *then there is* $p \in \mathcal{S}(\bigcup_{n<\omega} M_n)$ *such that* $n < \omega \Rightarrow p_n \leq p$.
(2) *If* $\langle M_i : i \leq \delta \rangle$ *is* $\leq_{\mathfrak{K}}$*-increasing continuous,* $p_i \in \mathcal{S}(M_i), (j < i \Rightarrow p_j \leq p_i)$, $p_i = tp(a_i, M_i, N_i)$ *and* $h_{i,j}$ *is a* $\leq_{\mathfrak{K}}$*-embedding of* N_j *into* N_i (*for* $j < i < \delta$) *such that* $h_{i,j} \restriction M_j = id_{M_j}, h_{i,j}(a_j) = a_i$, *then there is* $p_\delta \in \mathcal{S}(M_\delta), i \leq \delta \Rightarrow p_i \leq p_\delta$.
(3) *If* $\mathfrak{K}$ *has amalgamation in* λ *and is stable in* λ (*i.e.* $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$), *then*

(a) *every* $M \in K_\lambda$ *has a universal extension;*
(b) *for every* $M \in K_\lambda$ *and regular* $\theta \leq \lambda$ *there is* $N \in K_\lambda$ *which is* (λ, θ)*-saturated over* M.

Proof: Straightforward; similar to the proof of 0.26.

1. Weak diamond

1.1 Definition: Fix λ regular and uncountable.

(1)
$$\mathrm{WDmTId}(\lambda, S, \bar{\chi}) = \Big\{ A : A \subseteq \prod_{\alpha \in S} \chi_\alpha, \text{ and for some function } F \text{ with domain } \bigcup_{\alpha<\lambda} {}^{\alpha}(2^{<\lambda}) \text{ mapping } {}^{\alpha}(2^{<\lambda}) \text{ into } \chi_\alpha, \text{ for every } \eta \in A, \text{ for some } f \in {}^{\lambda}(2^{<\lambda}) \text{ the set } \{\delta \in S : \eta(\delta) = F(f \restriction \delta)\} \text{ is not stationary} \Big\}.$$

(Note: WDmTId stands for weak diamond target ideal.)[1]

Here we can replace $2^{<\lambda}$ by any set of this cardinality, and so we can replace $f \in {}^{\lambda}(2^{<\lambda})$ by $f_1, \dots, f_n \in {}^{\lambda}(2^{<\lambda})$ with F being n-place.

(2)

$$\mathrm{cov}_{\mathrm{wdmt}}(\lambda, S, \bar{\chi}) = \mathrm{Min}\Big\{|\mathcal{P}| : \mathcal{P} \subseteq \mathrm{WDmTId}(\lambda, S, \bar{\chi}) \text{ and } \prod_{\alpha<\lambda} \chi_\alpha \subseteq \bigcup_{A\in\mathcal{P}} A\Big\}.$$

(3)$\mathrm{WDmTId}_{<\mu}(\lambda, S, \bar{\chi}) = \Big\{A :$ for some $i^* < \mu$ and $A_i \in \mathrm{WDmTId}(\lambda, S, \bar{\chi})$ for $i < i^*$ we have $A \subseteq \bigcup_{i<i^*} A_i\Big\}.$

(4) $$\mathrm{WDmId}_{<\mu}(\lambda, \bar{\chi}) = \{S \subseteq \lambda : \mathrm{cov}_{\mathrm{wdmt}}(\lambda, S, \bar{\chi}) < \mu\}.$$

(5) Instead of "$< \mu^+$" we may write μ; if we omit μ we mean $(2^{<\lambda})$. If $\bar{\chi}$ is constantly 2 we may omit it; if $\chi_\alpha = 2^{|\alpha|}$ we may write pow instead of $\bar{\chi}$.

(6) $$\text{Let } \mu_{\mathrm{wd}}(\lambda, \bar{\chi}) = \mathrm{cov}_{\mathrm{wdmt}}(\lambda, \lambda, \bar{\chi}).$$

(7) We say that the **weak diamond** holds on λ if $\lambda \notin \mathrm{WDmId}(\lambda)$. We may omit $\bar{\chi}$ when it is constantly 1.

By [DvSh 65], [Sh b, XIV, 1.5, 1.10(2), 1.18(2), 1.9(2)] (presented better in [Sh f, AP,§1], note: 1.2(4) below relies on [Sh 460]) we have:

1.2 THEOREM:

(1) *If $\lambda = \aleph_1, 2^{\aleph_0} < 2^{\aleph_1}, \mu \leq (2^{\aleph_0})$ or even $2^\theta = 2^{<\lambda} < 2^\lambda, \mu = (2^\theta)^+$, or just: for some $\theta, 2^\theta = 2^{<\lambda} < 2^\lambda, \mu \leq 2^\lambda$, and $\chi^{<\lambda} < \mu$ for $\chi < \mu$, __then__ $\lambda \notin$ $WDmId_{<\mu}(\lambda)$. If in addition $(*)_{<\mu,\lambda}$ below holds, __then__ $\lambda \notin WDmId_{<\mu}(\lambda, pow)$, where:*

$(*)_{\mu,\lambda}$ there are no $A_i \in [\mu]^{\lambda^+}$ for $i < 2^\lambda$ such that $i \neq j \Rightarrow |A_i \cap A_j| < \aleph_0$ and $(*)_{<\mu,\lambda}$ means $(*)_{\chi,\lambda}$ holds for $\chi < \mu$.

(2) *If $\mu \leq \lambda^+$ or $cf([\mu_1]^{\leq\lambda}, \subseteq) < \mu$ for $\mu_1 < \mu$ & $cf(\mu) > \lambda$ or $\mu \leq (2^{<\lambda})^+$ __then__ $WDmId_{<\mu}(\lambda, \bar{\chi})$ is a normal ideal on λ. If this ideal is not trivial, __then__ $\lambda = cf(\lambda) > \aleph_0, 2^{<\lambda} < 2^\lambda$.*

(3) *A sufficient condition for $(*)_{<\mu,\lambda}$ is:*

(a) $\mu \leq 2^\lambda$ & $(\forall\alpha < \mu)(|\alpha|^{\aleph_0} < 2^\lambda)$.

(4) *Another sufficient condition for $(*)_{<\mu,\lambda}$ is:*

(b) $\mu \leq 2^\lambda$ & $\lambda \geq \beth_\omega$.

---

1 In [Sh b, AP,§1], [Sh f, AP,§1] we express $\mathrm{cov}_{\mathrm{wdmt}}(\lambda, S) > \mu^*$ by allowing $f(0) \in \mu^* < \mu$.

1.3 Remark: (1) So if $\mathrm{cf}(2^\lambda) < \mu$ (which holds if 2^λ is singular and $\mu = 2^\lambda$) then $(*)_{<\mu,\lambda}$ implies that there is $A \subseteq {}^\lambda 2, |A| < 2^\lambda, A \notin$ WDmTId(λ).
(2) Some related definitions appear in 1.13; we use them below (mainly $\mathrm{DfWD}_{<\mu}(\lambda)$), but as in a first reading it is recommended to ignore them, the definition is given later.
(3) We did not look again at the case $(\forall\sigma < \lambda)(2^\sigma < 2^{<\lambda} < 2^\lambda)$.

As in [Sh 88, 3.5], ([Sh 87a, 1.7], [Sh 87b, 6.3]):

1.4 Claim: *Assume $\lambda \notin$ $WDmId_{<\mu}(\lambda)$ or at least $DfWD_{<\mu}(\lambda)$ (where $\lambda = cf(\lambda) > \aleph_0$) and $\mathfrak{K}$ is an abstract elementary class.*
(1) *Assume $\mathfrak{K}$ is categorical in $\chi, \lambda = \chi^+$, and $\mathfrak{K}$ has a model in λ (if $LS(\mathfrak{K}) \leq \chi$ this is equivalent to: the model $M \in K_\chi$ is not $\leq_{\mathfrak{K}}$-maximal). Assume further $\mathfrak{K}$ does not have the amalgamation property in χ. Then for any $M_i \in \mathfrak{K}_\lambda$ for $i < i^* < \mu$, there is $N \in \mathfrak{K}_\lambda$ not $\leq_{\mathfrak{K}}$-embeddable into any M_i (and the assumptions of part (2) below holds).*
(2) *Assume $M_\eta \in K_{<\lambda}$ for $\eta \in {}^{\lambda>}2, M_\eta = \bigcup_{\alpha<\ell g(\eta)} M_{\eta\restriction(\alpha+1)}, \nu\triangleleft\eta \Rightarrow M_\nu \leq_{\mathfrak{K}} M_\eta$, and $M_{\eta\frown\langle 0\rangle}, M_{\eta\frown\langle 1\rangle}$ cannot be amalgamated over M_η (hence $M_\eta \neq M_{\eta\frown\langle\ell\rangle}$). Set $M_\eta =: \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$ for $\eta \in {}^\lambda 2$. Clearly M_η belongs to K_λ. For the $DfWD_{<\mu}(\lambda)$ version assume also*

- $(*)$ *$\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ is definable (even just by $\mathcal{L}_{\lambda,\lambda}$) in $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi, \mathfrak{K}_{<\chi}, \lambda, \mu)$.*

Then for any $N_i \in \mathfrak{K}_\lambda$ for $i < i^* < \mu$, there is $\eta \in {}^\lambda 2$ such that: M_η is not $\leq_{\mathfrak{K}}$-embeddable into any N_i.
(3) *In part (2), if $LS(\mathfrak{K}) \leq \lambda$ we can allow $N_i \in K_{\kappa_i}$ if $\sum_{i<i^*} cf([\kappa_i]^\lambda, \subseteq) < \mu$.*
(4) *Assume only $\lambda \notin$ $WDmId_{<\mu}(\lambda, \bar\chi)$. Part (2) holds if M_η is defined for $\eta \in \bigcup_{\alpha<\lambda}\prod_{i<\alpha}\chi_i$, and $\varepsilon < \zeta < \chi_i$, $\eta \in \prod_{j<i}\chi_j \Rightarrow M_{\eta\frown\langle\varepsilon\rangle}, M_{\eta\frown\langle\zeta\rangle}$ cannot be amalgamated over M_η.*
(5) *Similarly for a DfWD version. The assumption of Part (4) (hence the conclusion of Part (2)) holds if we assume that for $M \in K_\chi, i < \lambda$ there are χ_i $\leq_{\mathfrak{K}}$-extensions of M in K_λ, which pairwise cannot be amalgamated over M.*
(6) *Assume $\lambda = cf(\lambda)$, $2^\lambda > 2^{<\lambda} = 2^\theta$, and $\chi_i = 2^{(\theta)}$ and $\lambda \geq \beth_\omega$. Then $\lambda \in WDmId_{<2^\lambda}(\lambda, \bar\chi)$, hence if $< M_\eta : \eta \in \bigcup_\alpha \prod_{\beta<\alpha}\chi_\beta >$ are as in (4) then $|\{M_\eta \cong : \eta \in \prod_{\alpha<\lambda} M\chi_\alpha\}| = 2^\lambda$.*

Proof: (1) It is straightforward to choose $M_\eta \in K_\chi$ for $\eta \in {}^\alpha 2$ by induction on α, as required in part (2). Then use part (2) to get the desired conclusion.
(2) Without loss of generality the universe of N_i is λ and the universe of M_η is an ordinal γ_η such that $\eta \in {}^{\lambda>}2 \Rightarrow \gamma_\eta < \lambda$ and $\eta \in {}^\lambda 2 \Rightarrow \gamma_\eta = \lambda$. The

reader can ignore the "DfWD$_{<\mu}(\lambda)$" version (ignoring the h_η's, g) if he likes. For $\alpha < \lambda$ and $\eta \in {}^{\alpha}2$ let the function h_η be $h_\eta(i) = M_{\eta\restriction(i+1)}$ for $i < \ell g(\eta)$. Let $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ be the $<^*_\lambda$-first such object. For each $i < i^*$ we define $A_i \subseteq {}^{\lambda}2$ by $A_i = \{\eta \in {}^{\lambda}2 : M_\eta$ can be $\leq_{\mathfrak{K}}$-embedded into $N_i\}$.

For $\eta \in A_i$ choose f^i_η: $M_\eta \to N_i$, a $\leq_{\mathfrak{K}}$- embedding, hence $f^i_\eta \in {}^{\lambda}\lambda$. We also define a function F_i from $\bigcup_{\alpha<\lambda}({}^{\alpha}2 \times {}^{\alpha}\lambda)$ to $\{0,1\}$ by:

$F_i(\eta, f)$ is 0 <u>if</u> f is a $\leq_{\mathfrak{K}}$-embedding of M_η into N_i with range $\subseteq \ell g(\eta)$ which can be extended to a $\leq_{\mathfrak{K}}$-embedding of $M_{\eta\hat{}\langle 0\rangle}$ into N_i,

$F_i(\eta, f)$ is 1 otherwise.

Now for any $\eta \in A_i$, the set

$$E = \{\delta < \lambda : \gamma_{\eta\restriction\delta} = \delta \text{ and } f^i_\eta \restriction \delta \text{ is a function from } \delta \text{ to } \delta\}$$

is a club of λ. For every $\delta \in E$ clearly $F(\eta \restriction \delta, f^i_\eta \restriction \delta) = \eta(\delta)$ (as $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over M_η).

Hence (for the "Def" version see Definition 1.13(2) using 1.14(1), 1.14(3)) we have $A_i \in$ WDmTId$^{\rm Def}(\lambda)$. As $i^* < \mu$ clearly $\bigcup_{i<i^*} A_i \in$ WDmId$^{\rm Def}_{<\mu}(\lambda)$ and hence by assumption of the claim ${}^{\lambda}2 \neq \bigcup_i A_i$. Take $\eta \in {}^{\lambda}2 \setminus \bigcup_{i<i^*} A_i$. Then M_η is as required.

(3) Without loss of generality the universe of N_i is κ_i. Let $\mathcal{P}_i \subseteq [\kappa_i]^\lambda$ be a set of minimal cardinality such that $(\forall B)[B \subseteq \kappa_i \ \& \ |B| \leq \lambda \to (\exists B' \in \mathcal{P}_i)(B \subseteq B')]$. As $LS(\mathfrak{K}) \leq \lambda$ we can find for each $A \in \mathcal{P}_i$ a model $N^i_A \leq_{\mathfrak{K}} N_i$ of cardinality λ^+ whose universe includes A. Now apply part (2) to $\{N^i_A : i < i^*$ and $A \in \mathcal{P}_i\}$.

(4), (5) Same proof. $\blacksquare_{1.4}$

We give three variants of the preceding:

1.5 Claim: (1) *Assume*

$(*)_1$ $\lambda =$ *cf*$(\lambda) > \aleph_0$ *and*

(a) M_η *is a* τ*-model of cardinality* $< \lambda$ *for* $\eta \in {}^{\lambda>}2$ *and*

(b) *for each* $\eta \in {}^{\lambda}2$, $\langle M_{\eta\restriction\alpha} : \alpha < \lambda\rangle$ *is* $\subseteq$*-increasing continuous with union, called* $M_\eta \in K$, *of cardinality* λ;

(c) *if* $\eta \in {}^{\lambda>}2, \eta^\wedge\langle\ell\rangle \triangleleft \rho_\ell \in {}^{\lambda}2$ *for* $\ell = 1, 2$ *then* M_{ρ_1}, M_{ρ_2} *are not isomorphic over* M_η.

$(*)_2$ $\lambda \notin$ *WDmId*$_{<\mu}(\lambda)$.

<u>Then</u> $I(\lambda, K) \geq \mu$, *and in fact we can find* $X \subseteq {}^{\lambda}2$ *of cardinality* $\geq \mu$ *such that* $\{M_\rho : \rho \in X\}$ *are pairwise non-isomorphic.*

(2) *Assume*

$(*)_1^d$ *like* $(*)_1$ *in part* (1) *but in addition*

(d) $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ *is definable in* $\mathfrak{B} = \mathfrak{B}_\chi$.

$(*)_2^d$ $\lambda \notin DfWD_{<\mu}(\lambda)$.

Then the conclusion of part (1) *holds.*

(3) *The parallel of* 1.4(4) *holds.*

Proof: (1) Let $\{N_i : i < i^*\}$ be a maximal subset of $\{M_\rho : \rho \in {}^\lambda 2\}$ consisting of pairwise non-isomorphic models, and use the proof of 1.4(2) with $f_\eta : N_i \simeq M_\eta$. (2), (3) Left to the reader. $\blacksquare_{1.5}$

1.6 CLAIM: (1) *Assume*

$(*)_1$ $M_\eta \in K_{<\lambda}$ *for* $\eta \in {}^{\lambda>}2, \langle M_{\eta\restriction\alpha} : \alpha \le \ell g(\eta)\rangle$ *is* $\le_{\mathfrak{K}}$*-increasing continuous, and if* $\delta < \lambda$, $\eta^1\langle\ell\rangle \lhd v_\ell \in {}^\delta 2$ *for* $\ell = 0, 1$ *then* Mv_0, Mv_1 *are not isomorphic over* M_η *(or just the same for* $\delta = \lambda$*; by LS this is weaker).*

$(*)_2$ $\lambda =$ $cf(\lambda) > \aleph_0$, $\lambda \notin$ $WDmId_{<\mu}(\lambda)$, *and* λ *is a successor cardinal, or at least there is no* λ*-saturated normal ideal on* λ*, or at least* $WDmId(\lambda)$ *is not* λ*-saturated (which holds if for some* $\theta < \lambda, \{\delta < \lambda : cf(\delta) = \theta\} \notin WDmId(\lambda)$*).*

Then there is $A \subseteq {}^\lambda 2, |A| = 2^\lambda$ *such that: if* $\eta_1 \ne \eta_2$ *are in* A *then (taking* $M_\eta = \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$*)*

(α) $M_{\eta_1} \not\cong M_{\eta_2}$ *for*† $\eta_1 \ne \eta_2 \in A$ *and*

(β) *if* $(2^\chi)^+ < 2^\lambda$ *for* $\chi < \lambda$ *then we can also achieve:* M_{η_1} *cannot be* $\le_{\mathfrak{K}}$*-embedded into* M_{η_2}.

(2) *Under the assumptions of* 1.4(1) *we can find* $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ *as in the assumption of* 1.4(2).

(3) *Under the assumption of* 1.4(2) *the assumption of* 1.6(1) *holds.*

(4) *Under the assumption of* 1.6(1) *we have* $I(\lambda, \mathfrak{K}) = 2^\lambda$ *and if* $(2^\chi)^+ < 2^\lambda$ *then* $IE(\lambda, \mathfrak{K}) = 2^\lambda$.

(5) *The parallel of* 1.4(4) *holds (for non-isomorphism).*

Proof of 1.6: (1) The proof of [Sh 88, 3.5] works (see the implications preceding it). More elaborately, we divide the proof into cases according to the answer to the following:

† What about strengthening the result to "M_{η_1} is not $\le_{\mathfrak{K}}$-embeddable into M_{η_2}"? Even if we strengthen $(*)_1$ to:

$(*)_1'$ $M_{\eta^\wedge\langle 1\rangle}$ cannot be $\le_{\mathfrak{K}}$-embedded into M_v over M_η when $\eta^1\langle 0\rangle \trianglelefteq v \in {}^{\lambda>}2$

it will not help. Think of the case E^{M_η} is an equivalence relation with w^2 equivalence classes $\langle A_{\eta,n} : n < u\rangle$, $M_\eta = \sum_n (M_\eta \restriction A_{\eta,n})$ and each $M_\eta \restriction (\bigcup_n A_{\eta,n})$ is universal in $\mathfrak{K}_\lambda$.

Question: Is there $\eta^* \in {}^{\lambda>}2$ such that for every ν satisfying $\eta^* \trianglelefteq \nu \in {}^{\lambda>}2$ there are $\rho_0, \rho_1 \in {}^{\lambda>}2$ such that: $\nu \triangleleft \rho_0, \nu \trianglelefteq \rho_1$, and for any $v_0, v_1 \in {}^{\lambda}2$ satisfying $\rho_\ell \Delta \nu_\ell$ (for $\ell = 0, 1$), the models M_{v_0}, M_{v_1} are not isomorphic over M_η?

We can find a function $h : {}^{\lambda>}2 \to {}^{\lambda>}2$, such that:

(a) the function h is one-to-one, preserving $\triangleleft$ and $(h(\nu))\hat{}\langle\ell\rangle \trianglelefteq h(\nu\hat{}\langle\ell\rangle)$ (so for $v \in {}^{\lambda}2$ we let $h(v) = \bigcup_{\alpha<\lambda} h(v \restriction \alpha)$;

$(\mathrm{b})_{\mathrm{yes}}$ when the answer to the question is yes, it is exemplified by $\eta^* = h(\langle\rangle)$ and $M_{h(\nu\hat{}\langle 0\rangle)}, M_{h(\nu\hat{}\langle 1\rangle)}$ cannot be amalgamated over M_{η^*} (for every $\nu \in {}^{\lambda>}2$);

$(\mathrm{b})_{\mathrm{no}}$ when the answer to the question above is no, $h(\langle\rangle) = \langle\rangle, \ell < 2$ and if $\nu\hat{}\langle\ell\rangle \trianglelefteq \rho_0, \nu\hat{}\langle\ell\rangle \triangleleft \rho_1$ then $M_{h(\rho_0)}, M_{h(\rho_1)}$ are not isomorphic over $M_{h(\nu)}$.

Note that by transitivity of $\cong_{M_\eta}$, wlog $h(\eta)^\wedge\langle\ell\rangle \triangleleft h(\eta^\wedge\langle\ell\rangle)$. Without loss of generality h is the identity, by renaming (and we can preserve $(*)^d_1$ of 1.5(2) in the relevant case). Also clearly $M_{\eta\hat{}\langle\ell\rangle} \neq M_\eta$ (by the non-amalgamation assumption).

Case 1: The answer is yes. We do not use the non λ-saturation of WDmId(λ) in this case.

For any $\eta \in {}^{\lambda}2$ and $\leq_{\mathfrak{K}}$-embedding g of $M_{\langle\rangle}$ into $M_\eta =: \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$, let

$$A_{\eta,g} =: \{\nu \in {}^{\lambda}2 : \text{there is a } \leq_{\mathfrak{K}} \text{-isomorphism of } M_\nu \text{ onto } M_\eta \text{ extending } g\},$$

$$A_\eta =: \{\nu \in {}^{\lambda}2 : \text{there is a } \leq_{\mathfrak{K}} \text{-isomorphism of } M_\nu \text{ onto } M_\eta\}.$$

So: $|A_{\eta,g}| \leq 1$ for any g and $\eta \in A_\eta$ (as if $\nu_1, \nu_2 \in A_{\eta,g}$ are distinct then for some ordinal $\alpha < \lambda$ and $\nu \in {}^{\alpha}2$ we have $\nu =: \nu_0 \restriction \alpha = \nu_1 \restriction \alpha, \nu_0(\alpha) \neq \nu_1(\alpha)$ and use the choice of $h(\nu\hat{}\langle\ell\rangle)$).

Since $A_\eta = \bigcup\{A_{\nu,g} : g \text{ is a } \leq_{\mathfrak{K}} \text{-embedding of } M_{\langle\rangle} \text{ into } M_\eta\}$, we have $|A_\eta| \leq \lambda^{\|M_{\eta^*}\|} \leq 2^{<\lambda}$. Hence we can choose by induction on $\zeta < 2^\lambda, \eta_\zeta \in {}^{\lambda}2 \setminus \bigcup_{\xi<\zeta} A_{\eta_\xi}$ (existing by cardinality considerations as $2^{<\lambda} < 2^\lambda$). Then $\xi < \zeta \Rightarrow M_{\eta_\xi} \not\cong M_{\eta_\zeta}$, so we have proved clause (α).

Case 2: The answer is no.

Again, without loss of generality M_η has as universe the ordinal γ_η.

Let $\langle S_i : i < \lambda\rangle$ be a partition of λ to sets, none of which is in WDmId(λ). For each i we define a function F_i as follows:

if $\delta \in S_i, \eta, \nu \in {}^{\delta}2, \gamma_\eta = \gamma_\nu = \delta$, and $f \colon \delta \to \delta$ then

$F_i(\eta, \nu, f) = 0$ if we can find $\eta, \nu \in {}^{\lambda}2$ s.t. $\eta^1\langle 0\rangle \triangleleft \eta_1$ and $v^1\langle 0\rangle \triangleleft v_1$ s.t. f can be extended to an isomorphism of M onto M_{v_1},

$F_i(\eta, \nu, f) = 1$ otherwise.

So as $S_i \notin$ WDmId(λ), for some $\eta_i^* \in {}^{\lambda}2$ we have:

(∗) for every $\eta \in {}^{\lambda}2, \nu \in {}^{\lambda}2$ and $f \in {}^{\lambda}\lambda$ the following set of ordinals $i < \lambda$ is stationary:
$$\{\delta \in S_i : F_i(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = \eta^*_i(\delta)\}.$$

Now for any $X \subseteq \lambda$ let $\eta_X, \rho_X \in {}^{\lambda}2$ be defined by:
if $\alpha \in S_i$ then $i \in X \Rightarrow \eta_X(\alpha) = 1 - \eta^*_i(\alpha), i \notin X \Rightarrow \eta_X(\alpha) = 0$ and $\rho_X = \eta_{\{2i: i \in X\} \cup \{2i+1: i \notin X\}}$.

Now we show

$(*)'$ if $X, Y \subseteq \lambda$, and $X \neq Y$ then M_{ρ_X} is not $\leq_{\mathfrak{K}}$-isomorphic to M_{ρ_Y}.

Clearly $(*)'$ will suffice for finishing the proof.

Assume toward a contradiction that f is a $\leq_{\mathfrak{K}}$-isomorphism of M_{ρ_X} onto M_{ρ_Y}; as $X \neq Y$ there is i such that $i \in X \Leftrightarrow i \notin Y$ so there is $j \in \{2i, 2i+1\}$ such that $\rho_X \restriction S_j = \langle 1 - \eta^*_j(\alpha) : \alpha \in S_j \rangle$ and $\rho_Y \restriction S_j$ is identically zero. Clearly $E = \{\delta : f \text{ maps } \delta \text{ into } \delta\}$ is a club of λ and hence $S_j \cap E \neq \emptyset$.

So if $\delta \in S_j \cap E$ then f extends $f \restriction M_{\rho \restriction \delta}$ and is a $\leq_{\mathfrak{K}}$-isomorphism of it onto M_{ρ_Y}.

Now by the choice of F_j we get
$$\delta \in S_j \cap E \Rightarrow F_j(\rho_X \restriction \delta, \rho_Y \restriction \delta, f \restriction \delta) = \rho_X(\delta) = 1 - \eta^*_j(\delta).$$

But this contradicts the choice of η^*_j.

(2), (3), (4), (5). Check, similarly. $\blacksquare_{1.6}$

1.7 Conclusion: (1) Assume

$(*)_1$ for $\eta \in {}^{\lambda>}2, M_\eta \in K_{<\lambda}$ and $\langle M_{\eta\restriction\alpha} : \alpha \leq \ell g(\eta)\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $M_{\eta\hat{}\langle 1\rangle}$ cannot be $\leq_{\mathfrak{K}}$-embedded into M_ν over M_η when $\eta\hat{}\langle 0\rangle \trianglelefteq \nu \in {}^{\lambda>}2$ and $\langle M_\eta : \eta \in {}^{\lambda>}2\rangle$ is definable in $\mathfrak{B}$;

$(*)^d_2$ $\mathrm{WDmId}^{\mathrm{Def}}(\lambda)$ or $\mathrm{DfWD}^+(\lambda)$ is not λ-saturated (which holds if there is no normal λ-saturated ideals on λ (which holds for non-Mahlo λ) and holds if for some $\theta, \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is not in the ideal).

<u>Then</u> the conclusion of 1.6 holds.

From the Definition below, we use here mainly "superlimit".

1.8 Definition: (1) $M \in \mathfrak{K}_\lambda$ is a **superlimit** if

(a) for every $N \in \mathfrak{K}_\lambda$ satisfying $M \leq_{\mathfrak{K}} N$ there is $M' \in K_\lambda$ such that $N \leq_{\mathfrak{K}} M', N \neq M'$, and $M \cong M'$;

(b) if $\delta < \lambda^+$ is limit, $\langle M_i : i < \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing and $M_i \cong M$ (for $i < \delta$) then $\bigcup_{i<\delta} M_i \cong M$.

(2) For $\Theta \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda, \mu \text{ regular}\}$ we say $M \in \mathfrak{K}_\lambda$ is a (λ, Θ)-**superlimit** if:

(a) clause (α) from part (1) holds and
(b) if $M_i \cong M$ is ($\leq_{\mathfrak{K}}$)-increasing for $i < \mu \in \Theta$ then $\bigcup_{i<\mu} M_i \cong M$.

(3) For $S \subset \lambda^+$ we say $M \in \mathfrak{K}_\lambda$ is a (λ, S)-**strong limit** if:
(a) clause (a) from part (1) holds;
(b) there is a function F from $\bigcup_{\alpha<\lambda^+} {}^\alpha(K_\lambda)$ to K_λ such that:
(α) for any sequence $\langle M_i : i < \alpha \rangle$ if $\alpha < \lambda^+, M_0 = M, M_i$ is $\leq_{\mathfrak{K}}$-increasing, and $M_i \in \mathfrak{K}_\lambda$, <u>then</u> $j < \alpha \Rightarrow M_j \leq_{\mathfrak{K}} F(\langle M_i : i < \alpha \rangle)$,
(β) if $\langle M_i : i < \lambda^+ \rangle$ is $\leq_{\mathfrak{K}}$-increasing, $M_0 = M, M_i \in \mathfrak{K}_\lambda$, and for $i < \kappa, M_{i+1} \leq_{\mathfrak{K}} F(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{K}} M_{i+2}$ <u>then</u> $\{\delta \in S | \bigcup_{i<\delta} M_i \not\cong M\}$ is not stationary.

(4) M is a (λ, κ)-**limit** if there is a function F as in 3b(α) such that:
(a) if $\langle M_i : i < \kappa \rangle$ is a $<_{\mathfrak{K}}$-increasing continuous sequence in K_λ, $F(\bar{M} \restriction (i+1)) \leq_{\mathfrak{K}} M_{i+1}$ <u>then</u> $\bigcup_{i<\kappa} M_i \cong M$,
(b) there is at least one such sequence.

(5) M is a (λ, κ)-**superlimit** is defined similarly, but with F omitted and $M_{i+1} \cong M$.

1.9 Claim: (1) *In* 1.4(1) *we can replace the categoricity of $\mathfrak{K}$ in χ by "$\mathfrak{K}$ has a super limit model in χ" which is not an amalgamation base (see Definition* 1.8*). In this case the assumption of* 1.4(2), *and of* 1.6(1) *holds.*

(2) *We can weaken (in* 1.6(5)*) the existence of superlimit to "for some $\kappa = cf(\kappa) \leq \chi$ there is a $(\chi, \{\kappa\})$-superlimit model which is not an amalgamation base", provided that we add $\{\delta < \lambda : cf(\delta) = \kappa\} \notin$ WDmId$_{<\mu}(\lambda)$ (but for* 2.9(1) *we need now "WDmId$_{<\mu}(\lambda) + \{\delta < \lambda :\ cf(\delta) \neq \kappa\}$ is not λ-saturated". If there is $S \subseteq \{\delta < \lambda : cf(\delta) = \kappa\}$, which belongs to $I[\lambda]$ but not to WDmId$_{<\mu}(\lambda)$, we can weaken the model theoretic requirement to: there is a $(\chi, \{\kappa\})$-medium limit (see [Sh 88, Definition §3]) but not used here.*

1.10 Claim: *Assume* $2^\lambda < 2^{\lambda^+}$.

(0) *If $\mathfrak{K}$ (an abstract elementary class) is categorical in $\lambda, LS(\mathfrak{K}) \leq \lambda$, $I(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$, <u>then</u> $\mathfrak{K}_\lambda$ has amalgamation.*

(1) *If $\mathfrak{K}$ (an abstract elementary class) is categorical in λ, $LS(\mathfrak{K}) \leq \lambda$ and $1 \leq IE(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$ but $K_{\lambda^{++}} = \emptyset$ <u>then</u> $\mathfrak{K}$ has amalgamation in λ.*

(2) *Assume $\mathfrak{K}$ has amalgamation in $\lambda, LS(\mathfrak{K}) \leq \lambda, K_{\lambda^+} \neq \emptyset$ and $K_{\lambda^{++}} = \emptyset$. <u>Then</u> there is $M \in K_{\lambda^+}$ saturated above λ.*

(3) *If M is μ-saturated above $\lambda, LS(\mathfrak{K}) \leq \mu \leq \lambda_0 < \lambda$ and $\mathfrak{K}$ has amalgamation in every $\lambda_0' \in [\lambda_0, \lambda)$ <u>then</u> M is μ-saturated above λ_0.*

1.11 Remark: If $I(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$, then the assumption $\mathfrak{K}_{\lambda^{++}} = \emptyset$ is not used in part (1) of 1.10; this is 1.6(1) + (2). Also if $(2^\lambda)^+ < 2^{\lambda^+}$ then the assumption $K_{\lambda^{++}} = \emptyset$ is not needed in part (1) of 1.10; by 1.6(1) + (2) (note (b) of 1.6(1)).

Proof: (0) By 1.6(1) applied to λ^+.

(1) If not, we can choose for $\eta \in {}^{\lambda^+>}2$ a model $M_\eta \in \mathfrak{K}_\lambda$ such that $[\nu \triangleleft \eta \Rightarrow M_\nu \leq_{\mathfrak{K}} M_\eta]$, and $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over M_η. If $(2^\lambda)^+ < 2^{\lambda^+}$ we are done by 1.6(1), so assume $(2^\lambda)^+ = 2^{\lambda^+}$. For each $\eta \in {}^{\lambda^+}2$ let $M_\eta =: \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$, and let $N_\eta \in \mathfrak{K}_{\lambda^+}$ be such that $M_\eta \leq_{\mathfrak{K}} N_\eta, N_\eta$ is $\leq_{\mathfrak{K}}$-maximal (exists as $\mathfrak{K}_{\lambda^{++}} = \emptyset$). Now we choose by induction $\zeta < (2^\lambda)^+, \eta_\zeta \in {}^{\lambda^+}2$ such that M_{η_ζ} is not $\leq_{\mathfrak{K}}$-embeddable into N_{η_ξ} for $\xi < \zeta$ (exists by 1.4(2)). So necessarily for $\xi < \zeta, N_{\eta_\zeta}$ is not $\leq_{\mathfrak{K}}$-embeddable into N_{η_ξ} (as $M_{\eta_\zeta} \leq_{\mathfrak{K}} N_{\eta_\zeta}$). Also, for $\xi < \zeta, N_{\eta_\xi}$ is not $\leq_{\mathfrak{K}}$-embeddable into N_{η_ζ} as otherwise, by the maximality of N_{η_ξ}, this implies $N_{\eta_\xi} \cong N_{\eta_\xi}$. So $\{N_{\eta_\xi} : \zeta < 2^{\lambda^+}\}$ exemplifies $IE(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$, contradicting an assumption.

(2) A maximal model in K_{λ^+} will do by 0.20(2).

(3) Easy. $\blacksquare_{1.10}$

* * *

<u>1.12 Discussion</u>: Instead of Weak Diamond we now discuss Definable Weak Diamond, which is weaker but suffices.

Compare [MkSh 313], where many Cohen subsets are added to λ and a combinatorial principle about amalgamation of configurations $\langle M_s : s \subseteq n, s \neq n\rangle$ is obtained.

We are interested here in the case $n = 1$ (ordinary amalgamation); in §3, also $n = 2$, there even more definability can be required.

This is particularly interesting when we look at results under some other set theory; combining $2^\lambda = 2^{\lambda^+}$ with definable weak diamond on λ^+ is helpful. This played a major role in the preliminary try for this work.

1.13 Definition: (1) In Definition 1.1 we add the superscript $\mathcal{F}$ if we restrict ourselves to functions $F \in \mathcal{F}$.

(2) Fix a model $\mathfrak{B}$ whose universe includes λ and has a definable pairing function on λ, and a logic $\mathcal{L}$ closed under first order operations and substitution; also allow

"$M \in \mathfrak{K}$" and "$M \leq_{\mathfrak{K}} N$" in the formulas, if it is not said otherwise. Let

$$\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}} = \{F : \text{for some } g \in {}^{\lambda}\lambda \text{ and } \bar{h} = \langle h_\eta : \eta \in {}^{\lambda>}2\rangle \text{ where } h_\eta : \ell g(\eta) \to \lambda \text{ and } h_{\eta\restriction\beta} \subseteq h_\eta \text{ for } \beta < \ell g(\eta) \text{ and for some sequence } \bar{\psi} = \langle \psi_\alpha : \alpha < \lambda\rangle, \text{ with } \psi_\alpha \in \mathcal{L} \text{ for } \alpha < \lambda \text{ the following holds for every } \alpha < \lambda \text{ and } f \in {}^{\alpha}(2^{<\lambda}) : F(f) = 1 \text{ iff } (\mathfrak{B}, \alpha, g, h_f) \models \psi_\alpha \text{ iff } F(f) \neq 0\}.$$

(3) The version of weak diamond from 1.1, restricted to the class $\mathcal{F}$ of 1.13(2), is called the $(\mathfrak{B}, \mathcal{L})$-**definitional** version. If $\mathcal{L}$ is $\mathcal{L}_{\lambda,\lambda}$ we may omit it. If $\mathfrak{B}$ has the form† $(\mathcal{H}(\chi), \in, <^*_\chi, \lambda)$ we write $\mathcal{F}^{\mathrm{Def}(\chi)}_{\mathcal{L}}$ or $\mathcal{F}^{\mathrm{Def}(\chi)}$ instead of $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}}$ or $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B}}$, respectively. If we omit χ, we mean $\chi = (2^\lambda)^+$ and we may put $\mathrm{Def}(\chi)$ or Def instead of $\mathcal{F}^{\mathrm{Def}(\chi)}$ or $\mathcal{F}^{\mathrm{Def}}$ in the superscript. Having the definitional version or the **definable weak diamond** for λ means $\lambda \notin \mathrm{WDmId}^{\mathrm{Def}}(\lambda)$.

(4) Let $\mathrm{DfWD}_{<\mu}(\lambda)$ mean that with $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi, \lambda, \mu)$ we have $\lambda \notin \mathrm{WDmId}^{\mathrm{Def}}_{<\mu}(\lambda)$. Instead of "$< \mu^+$" we write "$\mu$" and instead of "$2^{<\lambda}$" we may write nothing.

(5) Let $\mathrm{DfWD}^+_{<\mu}(\lambda)$ mean $\mathrm{DfWD}_{<\mu}(\lambda)$ together with the principle $\bigotimes_\lambda$ below; we adopt the same conventions as in (4) concerning μ:

$\bigotimes_\lambda$ if for $\eta \in {}^{\lambda>}2, M_\eta$ is a $\tau_{\mathfrak{K}}$-model of cardinality $< \lambda, \langle M_{\eta\restriction\alpha} : \alpha \leq \ell g(\eta)\rangle$ is $\subseteq$-increasing continuous, for $\eta \in {}^{\lambda}2$ we let $M_\eta = \bigcup_{\alpha<\lambda} M_{\eta\restriction\alpha}$ and for $\eta \neq \nu \in {}^{\lambda}2, M_\eta$ and M_ν are not isomorphic over $M_{\langle\rangle}$, <u>then</u> $\{M_\eta/\cong : \eta \in {}^{\lambda}2\}$ has cardinality 2^λ (note that $2^\theta = 2^{<\lambda} < 2^\lambda$ implies that).

1.14 CLAIM:

(1) *Assume $\mathcal{L}$ first order or at least is definable enrichment of first order.*

In the definition of $\mathcal{F}^{\mathrm{Def}}_{\mathfrak{B},\mathcal{L}}$, we can replace "for every $\alpha < \lambda$" by "for a club of $\alpha < \lambda$". In the definition of $\mathcal{F}^{\mathrm{Def}}$ we can let $g \in {}^{\lambda}({}^{\lambda>}2)$ and $h_\eta : \ell g(\eta) \to {}^{\lambda>}2$. In any case $\mathit{WDmId}^{\mathcal{F}}_{<\mu}(\lambda)$ increases with $\mathcal{F}$ and is $\subseteq \mathit{WDmId}_{<\mu}(\lambda)$, similarly for $\mathit{WDmTId}^{\mathcal{F}}_{<\mu}(\lambda)$.

(2) *If $\mathcal{F} = \mathcal{F}^{\mathrm{Def}}_{\mathfrak{B}}$ and $M \leq (2^{<\lambda})^+$ or $(\forall \mu_\eta < \mu)(\text{if } ([\mu_1]^{\leq\lambda}, \subseteq) < \mu)$ & $cf(\mu) > \lambda$ <u>then</u> $\mathit{WDmId}^{F}_{<\mu}(\lambda)$ is a normal ideal (but possibly is equal to $\mathcal{P}(\lambda)$).*

(3) *Assume $V \models$ "$\lambda = \chi^+, \chi^{<\chi} = \chi, \mu > \lambda$" and P is the forcing notion of adding μ Cohen subsets to χ (i.e. $\{g : g$ a partial function from μ to $\{0\}$ with domain of cardinality $< \chi\}$). <u>Then</u> in V^P we have $\mathit{WDmId}^{\mathrm{Def}}_{<\mu}(\lambda)$ is the ideal of non-stationary subsets of λ; i.e. with $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\mu)^{V^P}$ for <u>any</u> χ. Also $\bigotimes_\lambda$ of Definition 1.13(5) holds.*

† We mean "for every such $\mathfrak{B}$" (but easily if $\underline{H}(\lambda^+) \in \underline{H}(\chi)$ it does not matter).

Remark: In 1.14(1) we use the assumption on $\mathcal{L}$; anyhow not serious: reread the definition 1.1(1).

Proof: (1), (2) By manipulating the h's (using the pairing function on λ).
(3) See [MkSh 313] or think. (The point is that we can break the forcing, first adding $\bar{\psi}$ and g (or the $< \mu$ ones) and then (read 1.1(1)) choose $\eta \in {}^{\lambda}2$ as $\underset{\sim}{g} \restriction [\gamma, \gamma + \lambda)$ not "used before". Now for any candidate $f \in {}^{\lambda}({}^{\lambda >}2)$ for a club of $\delta < \lambda, \eta(\delta) = g(\gamma + \delta)$ is not used in the definition of $f \restriction \delta, h_f \restriction \delta$ so stationarily often $\eta(\delta)$ "guesses" rightly.) $\blacksquare_{1.14}$

1.15 CLAIM: *(1) If $2^{\theta} = 2^{<\lambda} < 2^{\lambda}$ then $DfWD^+(\lambda)$.*
(2) $DfWD_{<\mu}(\lambda)$ holds when $\lambda \notin WDmId_{<\mu}(\lambda)$ (see 1.2 for sufficient conditions).

1.16 Discussion: We hope to get successful "guessing" not just on a stationary set, but on a positive set for the same ideal for which we have guessed; i.e. there is I a normal ideal on λ such that for $A \in I^+$ there is $\eta \in {}^{\zeta}2$ guessing I-positively; this is connected to questions on λ^+-saturation. For more see [Sh 638].

We phrased the following notion originally in the hope of later eliminating $\mu_{\text{wd}}(\lambda)$ (i.e. using 2^{λ} instead of $\mu_{\text{wd}}(\lambda)$).

1.17 Definition: (1)

$$\begin{aligned} \text{UDmId}^{\mathcal{F}}_{<\mu}(\lambda) = \{S \subseteq \lambda : & \text{ for some } i^* < \mu \text{ and } F_i \in \mathcal{F} \text{ (for } i < i^*) \\ & \text{for every } \eta \in {}^{S}2 \text{ there are } f \in {}^{\lambda}(2^{<\lambda}) \text{ and } i < i^* \\ & \text{and a club } E \text{ of } \lambda \text{ such that:} \\ & \text{for every } \delta \in E \text{ we have:} \\ & \delta \in S \Rightarrow \eta(\delta) = F_i(f \restriction \delta), \\ & \delta \in \lambda \setminus S \Rightarrow 0 = F_i(f \restriction \delta)\}. \end{aligned}$$

(2) We omit μ if $\mu = 1$.
(3) $BA^{\mathcal{F}}(\lambda)$ is defined as the family of $S \subseteq \lambda$ such that for some $F \in \mathcal{F}$ and $\eta = 1_S$ the condition above holds.

1.18 CLAIM: *Assume $\mathcal{F} = \mathcal{F}^{\text{Def}}_{\mathfrak{B}}$.*
(1) In the definition 1.17(1) we can replace $f \in {}^{\lambda}(2^{<\lambda})$ by $f \in {}^{\lambda}2$ or $f \in {}^{\lambda}({}^{\lambda >}2)$.
(2) $UDmId^{\mathcal{F}}(\lambda)$ is a normal ideal on λ (but possibly is $\mathcal{P}(\lambda)$).
(3) $BA^{\mathcal{F}}(\lambda)$ is a Boolean algebra of subsets of λ including all non-stationary subsets of λ and even $UDmId^{\mathcal{F}}(\lambda)$, and is closed under unions of $< \lambda$ sets and even under diagonal union.
(4) If $S \in BA^{\mathcal{F}}(\lambda)$ and $F \in \mathcal{F}$ then for some $\eta \in {}^{S}2$ we have:

(*) *for every* $f \in {}^{\lambda}({}^{\lambda>}2)$ *we have*

$$\{\delta \in S : \eta(\delta) = F(f \restriction \delta)\} \neq \emptyset \text{ mod } UDmId^{\mathcal{F}}(\lambda).$$

(5) $UDmId^{\mathcal{F}}_{<\mu}(\lambda) \subseteq WDmId^{\mathcal{F}}_{<\mu}(\lambda) \subseteq WDmId_{<\mu}(\lambda)$ *and they increase with* $\mathcal{F}$ *and* $\lambda \in UDmId^{\mathcal{F}}_{<\mu}(\lambda) \Leftrightarrow \lambda \in WDmId^{\mathcal{F}}_{<\mu}(\lambda)$.

Proof: Straightforward. $\blacksquare_{1.18}$

1.19 Discussion: Remember

$(*)_1$ If $V \models$ "$\chi = \chi^{<\chi}$ & $2^\chi = \chi^+$", $\mathbb{P}$ is the forcing notion of adding $\mu > \chi^+$ Cohen subsets to χ then in $\mathbf{V}^\chi$, any equivalence relation on $\mathcal{P}(\lambda)$ definable with parameters $X \subseteq \chi$ and ordinals which has at least χ^{++} equivalence classes has at least μ equivalence classes,

and (see [Sh 311], weaker see [Sh 237a]).

$(*)_2$ ZFC is consistent with $CH+$ for some stationary, costationary $S \subseteq \omega_1$ we have

(a) $\mathrm{WDmId}(\aleph_1) = \{A \subseteq \omega_1 : A \backslash S$ is not stationary$\}$,

(b) $\mathcal{D}_{\omega_1} + S$ is $\aleph_2$-saturated,

and (see [Sh 587]):

$(*)_3$ $ZFC + GCH$ is consistent with $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_1\} \in \mathrm{WDmId}(\aleph_2)$

and (see [Sh 208]):

$(*)_4$ $ZFC + 2^{\aleph_1} < 2^{\aleph_2}$ is consistent with $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\} \in \mathrm{WDmId}(\aleph_2)$.

See more on weak diamond [Sh 638].

2. First attempts

Given amalgamation in $\mathfrak{K}_\lambda$ (cf. 1.10(0)) we try to define and analyze types $p \in \mathcal{S}(M)$ for $M \in K_\lambda$. But types here (as in [Sh 300]) are not sets of formulas. They may instead be represented by triples (M, N, a) with $M \leq_{\mathfrak{K}} N$ and $a \in N \backslash M$. We look for "nice" types (i.e. triples) and try to prove mainly the density of the set of minimal types.

To simplify matters we allow uses of stronger assumptions than are ultimately desired (e.g. $2^{\lambda^+} > \lambda^{++}$ and/or $K_{\lambda^{+3}} = \emptyset$). These will later be eliminated. However the first extra assumption is still a "mild set theoretic assumption", and the second is harmless if we think only of proving our main theorem 0.2 and not on subsequent continuations.

So the aim of this section is to show that we can start to analyze such classes and introduce some basic notions: triples, minimal triples, reduced, the (weak) extension property.

2.1 HYPOTHESIS: $\mathfrak{K}$ is an abstract elementary class.

2.2 CLAIM: *Assume*

$(*)^2_\lambda$ *K is categorical in λ; $1 \le I(\lambda^+, K) < 2^{\lambda^+}$; $LS(\mathfrak{K}) \le \lambda$ and: $2^\lambda < 2^{\lambda^+}$, or at least the definable weak diamond holds for λ^+ holds.*

Then

(1) *$\mathfrak{K}_\lambda$ has amalgamation.*

(2) *If $I(\lambda^{++}, K) = 0$ then $\mathfrak{K}$ has a model in λ^+ which is universal homogeneous above λ, hence saturated above λ (see 0.22(2)).*

(3) *If $I(\lambda^{++}, K) = 0$ then $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \le \lambda^+$.*

Proof: (1) If amalgamation fails in K_λ and $2^\lambda < 2^{\lambda^+}$, then the assumptions of 1.4(1) hold with λ^+ in place of λ. Hence by 1.6(2) the statement $(*)_1$ of 1.6(1) (see there) holds and easily also $(*)_2$ of 1.6(1), hence by 1.6(4) we have $I(\lambda^+, K) = 2^{\lambda^+}$, a contradiction. If $2^\lambda = 2^{\lambda^+}$, we are using the variants from 1.13.

(2) As $I(\lambda^{++}, K) = 0 < I(\lambda^+, K)$, there is $M \in K_{\lambda^+}$ which is maximal. If M is not universal homogeneous above λ then there are $N_0, N_1 \in K_\lambda$ with $N_0 \le_{\mathfrak{K}} M$ and $N_0 \le_{\mathfrak{K}} N_1$ such that N_1 cannot be $\le_{\mathfrak{K}}$-embedded into M over N_0. Use 0.20(2) to get a contradiction.

(3) Follows from (2). $\blacksquare_{2.2}$

2.3 Definition:

(1) (a) $K^3_\lambda = \{(M_0, M_1, a) : M_0 \le_{\mathfrak{K}} M_1$ are both in K_λ and $a \in M_1 \backslash M_0\}$.

(b) $(M_0, M_1, a) \le (M_0', M_1', a')$ if $a = a', M_0 \le_{\mathfrak{K}} M_0', M_1 \le_{\mathfrak{K}} M_0'$.

(c) $(M_0, M_1, a) \le_h (M_0', M_1', a')$ if $h(a) = a'$, and for $\ell = 0, 1$ we have: $h \restriction M_\ell$ is a $\le_{\mathfrak{K}}$-embedding of M_ℓ into M_ℓ'.

(d) $(M_0, M_1, a) < (M_0', M_1', a')$ if $(M_0, M_1, a) \le (M_0', M_1', a)$ and $M_0 \ne M_0'$.

(e) Similarly $<_h$.

(2) $(M_0, M_1, a) \in K^3_\lambda$ has the **weak extension property** if there is $(M_0', M_1', a) \in K^3_\lambda$ such that $(M_0, M_1, a) \le (M_0', M_1', a)$ and $M_0 \ne M_0'$.

(3) $(M_0, M_1, a) \in K^3_\lambda$ has the **extension property** if: for every $N_0 \in \mathfrak{K}_\lambda$ and $\le_{\mathfrak{K}}$-embedding f of M_0 into N_0 there are N_1, b and g such that: $(M_0, M_1, a) \le_g (N_0, N_1, b) \in K^3_\lambda$ and $g \supseteq f$ (so $g(a) = b$ and g is a $\le_{\mathfrak{K}}$-embedding of M_1 into N_1).

2.4 CLAIM: *Assume*
$(*)^3_\lambda$ $LS(\mathfrak{K}) \leq \lambda, K$ *is categorical in* λ *and in* λ^+, *and* $1 \leq I(\lambda^{++}, K)$.
Then *every* $(M_0, M_1, a) \in K^3_\lambda$ *has the weak extension property, that is:*
if $M_0 \leq_{\mathfrak{K}} M_1$ *are in* K_λ *and* $a \in M_1 \backslash M_0$, *then we can find* M'_0, M'_1 *in* $\mathfrak{K}_\lambda$ *such that:* $M_0 <_{\mathfrak{K}} M'_0$ *hence* $M_0 \neq M'_0$ *and* $(M_0, M_1, a) \leq (M'_0, M'_1, a)$.

Proof: We can choose $\langle N_i, a_i : i < \lambda^+ \rangle$ such that:

(a) $N_i \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuous in i;

(b) h_i is an isomorphism from M_1 onto N_{i+1} such that $h_i(M_0) = N_i, h_i(a) = a_i$.

Now as $a \in M_1 \backslash M_0$ clearly $i < j < \lambda^+ \Rightarrow a_i \in N_{i+1} \leq_{\mathfrak{K}} N_j$ and $a_j \notin N_i$ hence $\bigcup_{i<\lambda^+} N_i \in K_{\lambda^+}$.

By 0.20(1) applied to λ^+ there are $M'_0 \leq_{\mathfrak{K}} M'_1$ in $K_{\lambda^+}, M'_0 \neq M'_1$, and there is $b \in M'_1 \backslash M'_0$. As K is categorical in λ^+, without loss of generality $M'_0 = \bigcup_{i<\lambda^+} N_i$.

Let χ be large enough and $\mathfrak{B} \prec (\mathcal{H}(\chi) \in, <^*_\chi)$ be such that $\lambda \subseteq \mathfrak{B}, \|\mathfrak{B}\| = \lambda$ and $\langle N_i, a_i : i < \lambda^+ \rangle, M'_0, M'_1, b$ and the definition of $\mathfrak{K}$ belong to $\mathfrak{B}$.

Let $\delta = \mathfrak{B} \cap \lambda^+$, so $\delta \in (\lambda, \lambda^+)$ is a limit ordinal and

$$\begin{gathered} N_\delta \leq_{\mathfrak{K}} N_{\delta+1} \leq_{\mathfrak{K}} M'_1, \\ N_\delta \leq_{\mathfrak{K}} (M'_1 \cap \mathfrak{B}) \leq_{\mathfrak{K}} M'_1, \\ \mathfrak{B} \cap M'_0 = N_\delta, \\ N_{\delta+1} \cap (M'_1 \cap \mathfrak{B}) = N_\delta, \end{gathered}$$

so for some N we have:

$$N \in \mathfrak{K}_\lambda, N \leq_{\mathfrak{K}} M'_1, \text{ and } (N_{\delta+1} \cup (M'_1 \cap \mathfrak{B})) \subseteq N$$

so (see Definition 2.3(1) above)

$$(N_\delta, N_{\delta+1}, a_\delta) \leq_{\mathfrak{K}} (M'_1 \cap \mathfrak{B}, N, a_\delta),$$

and b witnesses that $N_\delta \neq M'_1 \cap \mathfrak{B}$.

As $(M_0, M_1, a) \cong (N_\delta, N_{\delta+1}, a_\delta)$, the result follows. $\blacksquare_{2.4}$

2.5 Definition: (1) $(M_0, M_1, a) \in K^3_\lambda$ is **minimal** when:

if $(M_0, M_1, a) \leq_{h_\ell} (M'_0, M^\ell_1, a_\ell) \in K^3_\lambda$ for $\ell = 1, 2$,
and $h_1 \restriction M_0 = h_2 \restriction M_0$
then $\text{tp}(a_1, M'_0, M^1_1) = \text{tp}(a_2, M'_0, M^2_1)$.

(2) $(M_0, M_1, a) \in K^3_\lambda$ is **reduced** when:

$$\text{if } (M_0, M_1, a) \leq (M'_0, M'_1, a) \in \mathfrak{K}^3_\lambda \text{ then } M'_0 \cap M_1 = M_0.$$

(3) We say $p \in \mathcal{S}(M_0)$ is **minimal**, where $M_0 \in K_\lambda$, if for some a, M_1 we have: $p =$ tp(a, M_0, M_1) and $(M_0, M_1, a) \in K^3_\lambda$ is minimal.
(4) We say $p \in \mathcal{S}(M_0)$ is **reduced** where $M_0 \in K_\lambda$, if for some a, M_1 we have $p =$ tp(a, M_0, M_1) and $(M_0, M_1, a) \in K^3_\lambda$ is reduced.
(5) We say $p \in \mathcal{S}(M)$ where $M \in K_\lambda$, is **algebraic** if for no $c \in M$ is $p =$ tp(c, M, M).

2.6 FACT: (1) For every $(M_0, M_1, a) \in K^3_\lambda$ there is a reduced (M'_0, M'_1, a) such that: $(M_0, M_1, a) \leq (M'_0, M'_1, a) \in K^3_\lambda$.
(2) Assume $\langle (M_{0,\alpha}, M_{1,\alpha}, a) : \alpha < \delta \rangle$ is an increasing sequence of members of K^3_λ.

- (a) if $\delta < \lambda^+$ <u>then</u>† $(M_{0,\alpha}, M_{1,\alpha}, a) \leq (\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a) \in K^3_\lambda$ for $\alpha < \delta$.
- (b) If $\delta = \lambda^+$ the result may be in $K^3_{\lambda^+}$: if $\{\alpha < \delta : M_{0,\alpha} \neq M_{0,\alpha+1}\}$ is cofinal, this holds.
- (c) If $\delta < \lambda^+$ and each $(M_{0,\alpha}, M_{1,\alpha}, a)$ is reduced <u>then</u> so is $(\bigcup_{\beta<\delta} M_{0,\beta}, \bigcup_{\beta<\delta} M_{1,\beta}, a)$.

(3) If $(M_0, M_1, a) \leq (M'_0, M'_1, a)$ are in K^3_λ and the first triple is minimal <u>then</u> so is the second.
(4) If $(M_0, M_1, a) \leq (M'_0, M'_1, a)$ are in K^3_λ <u>then</u> tp$(a, M_0, M_1) \leq$ tp(a, M'_0, M'_1) (see Definition 0.19(6)).
(5) If K_λ has amalgamation, then: $(M_0, M_1, a) \in K^3_\lambda$ is minimal if and only if:

- (∗) If $(M_0, M_1, a) \leq_{h_\ell} (M'_0, M'_1, a_\ell) \in K^3_\lambda$ for $\ell = 1, 2$ and $h_1 \restriction M_0 = h_2 \restriction M_0$ <u>then</u> tp$(a_1, M'_0, M'_1) =$ tp(a_2, M'_0, M'_1).

(6) If there is no maximal member[2] of K^3_λ and there are $N_0 <_{\mathfrak{K}} N_1$ in K_λ, <u>then</u> there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$.
(7) If every triple in K^3_λ has the weak extension property, and there are $N_0 <_{\mathfrak{K}} N_1$ in $\mathfrak{K}_\lambda$, <u>then</u> there are $N^0 <_{\mathfrak{K}} N^1$ in $\mathfrak{K}_{\lambda^+}$.
(8) If LS$(\mathfrak{K}) \leq \lambda$ and every triple in K^3_λ has the extension property and $K^3_\lambda \neq \emptyset$ <u>then</u> no $M \in K_{\lambda^+}$ is $<_{\mathfrak{K}}$-maximal hence $K_{\lambda^{++}} \neq \emptyset$.
(9) If $LS(\mathfrak{K}) \leq \lambda$ and $K_{\lambda^+} \neq \emptyset$, <u>then</u> $K^3_\lambda \neq \emptyset$.
(10) Assume K_λ has amalgamation. If $(M_0, M_1, a) \in K^3_\lambda$ and $p =$ tp(a, M_0, M) is minimal, <u>then</u> (M_0, M_1, a) is a minimal triple (i.e. in Definition 2.5(3) we

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† If we deal with an increasing sequence of types, the existence of univ is not clear.

2 This will be applied for λ^+.

can replace 'for some' by 'for all'. Also, p is minimal iff for no N do we have $M_0 \leq_{\mathfrak{K}} N \in K$ and p has more than one non-algebraic extension in $\mathcal{S}(N)$.

Proof: Easy. Note that part (7) is 2.4.

2.7 CLAIM: *Assume $(*)^2_{\lambda^+} + (*)^3_\lambda$ (i.e. the hypothesis of 2.2 and 2.4) and $2^{\lambda^+} > \lambda^{++}$, and $K_{\lambda^{+3}} = \emptyset$.*

Then in $\mathfrak{K}^3_\lambda$ the minimal triples are dense (i.e. above every triple in K^3_λ there is a minimal one).

Remark: We do not intend to adopt the hypotheses "$2^{\lambda^+} > \lambda^{++}$", $K_{\lambda^{+3}} = \emptyset$ indefinitely. They will be eliminated in §3.

Proof: If not, we can choose by induction on $\alpha < \lambda^+$, for $\eta \in {}^{\alpha}2$ a triple $(M^0_\eta, M^1_\eta, a_\eta)$ and $h_{\eta,\nu}$ for $\nu \trianglelefteq \eta$ such that:

(i) $(M^0_\eta, M^1_\eta, a_\eta) \in K^3_\lambda$,

(ii) $\nu \triangleleft \eta \Rightarrow (M^0_\nu, M^1_\nu, a_\nu) \leq_{h_{\eta,\nu}} (M^0_\eta, M^1_\eta, a_\eta)$,

(iii) $\nu_0 \triangleleft \nu_1 \triangleleft \nu_2 \Rightarrow h_{\nu_2,\nu_0} = h_{\nu_2,\nu_1} \circ h_{\nu_1,\nu_0}$,

(iv) $(M^0_{\eta\hat{}\langle\ell\rangle}, M^1_{\eta\hat{}\langle\ell\rangle}, h_{\eta\hat{}\langle\ell\rangle,\eta} \restriction M^0_\eta)$ for $\ell = 0, 1$ are equal,

(v) $\mathrm{tp}(a_{\eta\hat{}\langle 0\rangle}, M^0_{\eta\hat{}\langle 0\rangle}, M^1_{\eta\hat{}\langle 0\rangle}) \neq \mathrm{tp}(a_{\eta\hat{}\langle 1\rangle}, M^0_{\eta\hat{}\langle 1\rangle}, M^1_{\eta\hat{}\langle 1\rangle})$; this makes sense as $M^0_{\eta\hat{}\langle 0\rangle} = M^0_{\eta\hat{}\langle 1\rangle}$,

(vi) if $\eta \in {}^{\delta}2$ and $\delta < \lambda^+$ is a limit ordinal, then $M^\ell_\eta = \bigcup_{\alpha<\delta} h_{\eta,\eta\restriction\alpha}(M^\ell_{\eta\restriction\alpha})$ for $\ell = 0, 1$,

(vii) $(M^0_{<>}, M^1_{<>}, a_{<>}) \in K^3_\lambda$ is a triple above which there is no minimal one.

This is straightforward: for $\alpha = 0$ choose a triple in K^3_λ above which supposedly there is no minimal triple; in limit α take limits of diagrams (chasing the h's); in successor α, use non-minimality and 2.6(5).

Let $M^* \in \mathfrak{K}_{\lambda^{++}}$ be saturated above λ (exists by 2.2(2) so it is necessarily homogeneous universal above λ^+, hence above λ; note: λ there stands for λ^+ here).

We choose by induction on $\alpha < \lambda^+$ for $\eta \in {}^{\alpha}2$, a $\leq_{\mathfrak{K}}$-embedding g_η of M^0_η into M^* such that:

$$\nu \triangleleft \eta \Rightarrow g_\nu = g_\eta \circ h_{\eta,\nu},$$

$$g_{\eta\hat{}\langle 0\rangle} = g_{\eta\hat{}\langle 1\rangle}.$$

This is clearly possible. Let $N^0_\eta = M^* \restriction \mathrm{Rang}(g_\eta)$. For $\eta \in {}^{\lambda^+}2$ let $N^0_\eta = M^* \restriction \bigcup_{\alpha<\lambda^+} \mathrm{Rang}(g_{\eta\restriction\alpha})$ and let $g_\eta = \bigcup_{\alpha<\lambda^+} g_{\eta\restriction\alpha}$. Chasing arrows we can find for $\eta \in {}^{\lambda^+}2$ a limit to $(\langle M^0_{\eta\restriction\alpha}, M^1_{\eta\restriction\alpha}, a_{\eta\restriction\alpha}\rangle, h_{\eta\restriction\beta,\eta\restriction\alpha} : \alpha < \beta < \lambda^+)$, say $(M^0_\eta, M^1_\eta, a_\eta) \in K^3_{\lambda^+}$, and $h_{\eta,\nu}$ for $\nu \triangleleft \eta$ as usual. Let f_η be the function from M^0_η

into M^* such that for $\alpha < \lambda^+$ we have $f_\eta \circ h_{\eta,\eta\restriction\alpha} = g_\eta$. So f_η is a $\leq_{\mathfrak{K}}$-embedding of M^0_η into M^*. So we can extend f_η to f^+_η, a $\leq_{\mathfrak{K}}$-embedding of M^1_η into M^*.

Let $a^*_\eta = f^+_\eta(a_\eta)$ for $\eta \in {}^{\lambda^+}2$.

As $2^{\lambda^+} > \lambda^{++}$ for some $\eta_0 \neq \eta_1$ we have $a^*_{\eta_0} = a^*_{\eta_1}$. So for some $\alpha < \lambda^+$, $\eta_0 \restriction \alpha = \eta_1 \restriction \alpha$ but $\eta_0(\alpha) \neq \eta_1(\alpha)$, without loss of generality $\eta_\ell(\alpha) = \ell$ and by clause (v) above we get a contradiction. $\blacksquare_{2.7}$

2.8 Claim: (1) *Assume* $(*)^2_\lambda$ *or just*

$(*)^{2^-}_\lambda$ *$\mathfrak{K}$ has amalgamation in λ and $LS(\mathfrak{K}) \leq \lambda$.*

If $M_0 \leq_{\mathfrak{K}} N_0 \in K_\lambda$ and $(M_0, M_1, a) \in K^3_\lambda$ then there is $N \in K_{\leq\lambda^+}$ such that: $N_0 \leq_{\mathfrak{K}} N$ and for every $c \in N$ satisfying $tp(c, M_0, N) = tp(a, M_0, M_1)$, there is a $\leq_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} such that $h(a) = c$ and $N \notin K_{\lambda^+} \Rightarrow N$ is a $<_{\mathfrak{K}}$-maximal member of K_λ.

(2) Assume $M_0 \leq_{\mathfrak{K}} N_0 \in K_\lambda$ and $(M_0, M_1, a) \in K^3_\lambda$ and every triple in K^3_λ has the weak extension property. Then there is $N \in K_{\lambda^+}$ such that: $N_0 \leq_{\mathfrak{K}} N$ and for every $c \in N$ either for some $N' \in K_\lambda$ we have $N_0 \cup \{c\} \subseteq N' \leq_{\mathfrak{K}} N$ and c does not strongly realize $tp(a, M_0, M_1)$ or there is an $\leq_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} such that $h(a) = c$.

(3) We can in parts (1), (2) have $N_0 \in K_{\lambda^+}$.

Proof: (1) We choose, by induction on $\alpha < \lambda^+$, a model $N_\alpha \in K_\lambda$ increasing (by $\leq_{\mathfrak{K}}$) continuous such that: for α even $N_\alpha \neq N_{\alpha+1}$ if N_α is not $\leq_{\mathfrak{K}}$-maximal, and for α odd let $\beta_\alpha = \text{Min}\{\beta : \beta = \alpha + 1$ or $\beta \leq \alpha$ and there is $c \in N_\beta$ such that there is no $\leq_{\mathfrak{K}}$-embedding h of M_1 into N_α extending id_{M_0} such that $h(a) = c$ but for some $N \in K_\lambda, N_\alpha \leq_{\mathfrak{K}} N$ and there is a $\leq_{\mathfrak{K}}$-embedding h of M_1 into N extending id_{M_0} such that $h(a) = c\}$, and if $\beta_\alpha \leq \alpha$ then choose N exemplifying this and let $N_{\alpha+1} = N$. By the definition of type we are done.

(2) Same proof; note that the non $\leq_{\mathfrak{K}}$-maximality of N_α (and hence N) follows by a weak extension property.

(3) By using 0.20(2) repeatedly λ^+ times. $\blacksquare_{2.8}$

2.9 Claim: *Assume* $(*)^2_\lambda$ *or just:*

$(*)^{2^-}_\lambda$ *$\mathfrak{K}$ has amalgamation for λ and $LS(\mathfrak{K}) \leq \lambda$.*

(1) Assume that above $(M_0, M_1, a) \in K^3_\lambda$ there is no minimal member of K^3_λ; then (M_0, M_1, a) itself has the extension property.

(2) *If* $(M_0, M_1, a) \in K^3_\lambda, M_0 \leq_{\mathfrak{K}} N \in K$ *and the number of* $c \in N$ *such that* $tp(c, M_0, N) = tp(a, M_0, M_1)$ *is* $> \lambda$ *<u>then</u>* (M_0, M_1, a) *has the extension property.*
(3) *Assume above* $(M_0, M_1, a) \in K^3_\lambda$ *there is no minimal member of* K^3_λ; *<u>then</u>*
 $(*)$ *for some* N *we have:* $M_0 \leq_{\mathfrak{K}} N$ *and* N *is as required in part* (2).

2.10 Remark: (1) See 2.3(3).
(2) Note that $(*)^2_\lambda$ is from 2.2 and $(*)^2_\lambda \Rightarrow (*)^{2^-}_\lambda$ by 2.2.

Proof: (1) Follows by part (2) and (3).
(2) By 0.12(1)(D). Without loss of generality N has cardinality λ^+ and also is as in 2.8. By 0.20(2) for any M_0' such that $M_0 \leq_{\mathfrak{K}} M_0' \in K_\lambda$ there is $N_1, N \leq_{\mathfrak{K}} N_1 \in K_{\lambda^+}$ and a $\leq_{\mathfrak{K}}$-embedding h of M_0' into N_1 extending id_{M_0}. Now some $c \in N \backslash h(M_0')$ realizes $\mathrm{tp}(a, M_0, M_1)$ so (by the use of 2.8) there is an embedding h_c of M_1 into N extending id_{M_0} such that $h_c(a) = c$. Lastly let $N_1' \leq_{\mathfrak{K}} N_1$ be of cardinality λ and including $\mathrm{Rang}(h_c) \cup \mathrm{Rang}(h)$ (send a to c via h_c). So modulo chasing arrows we have proved that (M_0, M_1, a) has the extension property for the case $M_0' \in K_\lambda$, $M_0 \leq_{\mathfrak{K}} M_0'$, which was arbitrary so we are done.
(3) We first prove
 $(*)_0$ For some $M_0^+, M_0 \leq_{\mathfrak{K}} M_0^+ \in K_\lambda$ and $\mathrm{tp}(a, M_0, M_1)$ has $> \lambda$ extensions in $\mathcal{S}(M_0^+)$ (in fact $\geq \min\{2^\mu : 2^\mu > \lambda\}$).

Proof of $(*)_0$*:* Let $M^0_\eta, M^1_\eta, a_\eta, h_{\eta,\nu}$ ($\eta \in {}^{\lambda^+ >}2$ and $\nu \trianglelefteq \eta$) be as in the proof of 2.7 (i.e. satisfy (i)–(vi) there) and $M^0_{<>} = M_0$, $M^1_{<>} = M_1$. Let $\mu = \mathrm{Min}\{\mu : 2^\mu > \lambda\}$, so ${}^{\mu>}2$ has cardinality $\leq \lambda$ and $\mu \leq \lambda$. Let ${}^{\mu>}2 = \{\eta_\zeta : \zeta < \zeta^*\}$ be such that $\eta_\xi \triangleleft \eta_\zeta \Rightarrow \xi < \zeta$ and so $\zeta^* < \lambda^+$ and without loss of generality is a limit ordinal. Now we can choose by induction on $\zeta \leq \zeta^*$ a model $M^*_\zeta \in K_\lambda$ and, if $\zeta < \zeta^*$, also a function g_{η_ζ} such that:
 (α) M^*_ζ is $\leq_{\mathfrak{K}}$-increasing continuous in ζ,
 (β) $M^*_0 = M_{\langle\rangle} = M_0$,
 (γ) g_{η_ζ} is a $\leq_{\mathfrak{K}}$-embedding of $M^0_{\eta_\zeta}$ into $M^*_{\zeta+1}$,
 (δ) if $\eta_\xi \triangleleft \eta_\zeta$ then $g_{\eta_\zeta} \circ h_{\eta_\zeta, \eta_\xi} = g_{\eta_\xi}$,
 (ε) if $\xi < \xi_0, \xi < \xi_1, \eta_{\xi_0} = \eta_\xi{}^\frown\langle 0\rangle, \eta_{\xi_1} = \eta_\xi{}^\frown\langle 1\rangle$ then ($M^*_{\xi_0} = M^*_{\xi_1}$ and) $g_{\eta_{\xi_0}} = g_{\eta_{\xi_1}}$.

So for $\eta \in {}^\mu 2$ we can find g_η, a $\leq_{\mathfrak{K}}$-embedding of M^0_η into $M^*_{\zeta^*}$, such that $g_\eta \circ h_{\eta, \eta\restriction\alpha} = g_{\eta\restriction\alpha}$ for every $\alpha < \mu$. We also can let

$$p^0_\eta = g_\eta[\mathrm{tp}(a_\eta, M^0_\eta, M^1_\eta)] \in \mathcal{S}(M^*_{\zeta^*} \restriction \mathrm{Rang}(g_\eta)),$$

and find p_η such that $p^0_\eta \leq p_\eta \in \mathcal{S}(M^*_{\zeta^*})$ (possible as $\mathfrak{K}_\lambda$ has amalgamation by 2.2(1) if $(*)^2_\lambda$ holds and by $(*)^{2^-}_\lambda$ otherwise).

For $\eta \in {}^{\mu}2$ and $\alpha \leq \mu$ let $N^0_{\eta\restriction\alpha} = M^*_{\zeta^*} \restriction \mathrm{Rang}\,(g_{\eta\restriction\alpha})$. Clearly for $\eta \in {}^{\mu \geq}2, N^0_\eta$ is well defined; $\eta \triangleleft \nu \in {}^{\mu\geq}2 \Rightarrow N^0_\eta \leq_{\mathfrak{K}} N^0_\nu$; and $N^0_{\eta\hat{}\langle 0\rangle} = N^0_{\eta\hat{}\langle 1\rangle}$. Also letting, for $\eta \in {}^{\mu}2$ and $\alpha \leq \mu$, the type $p^0_{\eta\restriction\alpha}$ be $p^0_\eta \restriction N^0_{\eta\restriction\alpha}$ we have: $p^0_\eta \in \mathcal{S}(N^0_\eta)$ is well defined, $\eta \triangleleft \nu \in {}^{\mu\geq}2 \Rightarrow p^0_\eta \leq p^0_\nu$ and $p^0_{\eta\hat{}\langle 0\rangle} \neq p^0_{\eta\hat{}\langle 1\rangle}$. Hence for $\eta_0 \neq \eta_1$ from ${}^{\mu}2$ we have $p_{\eta_0} \neq p_{\eta_1}$. So $|\{q \in \mathcal{S}(M^*_{\zeta^*}) : q \restriction M_0 = p\}| \geq 2^\mu > \lambda$. Let $M^+_0 = M^*_{\xi^*}$.

Proof of $(*)$: We choose by induction on $i < \lambda^+, N_i \in K_\lambda$ which is $\leq_{\mathfrak{K}}$-increasing continuous, $N_0 = M^+_0$ (M^+_0 is from $(*)_0$ above) and for each i some $c_i \in N_{i+1}$ realizes over $N_0 = M^+_0$ a (complete) extension of $p = \mathrm{tp}(a, M_0, M_1)$ not realized in N_i. There is such a type by clause $(*)_0$ above and there is such an N_{i+1} as $\mathfrak{K}$ has amalgamation in λ. Clearly $c_i \notin N_i$ and so $\bigcup_{i<\lambda^+} N_i$ is as required. $\blacksquare_{2.9}$

2.11 Claim: *Assume* $(*)^{2^-}_\lambda$ *(from 2.9; that is* $\mathfrak{K}$ *has amalgamation in* λ *and* $LS(\mathfrak{K}) \leq \lambda$*).*

If $(M_0, M_1, a) \leq (M'_0, M'_1, a)$ *are from* K^3_λ*, and the second has the extension property,* then *so does the first.*

Proof: Use amalgamation over M_0: if $M_0 \leq_{\mathfrak{K}} N_0 \in K_\lambda$ we can find N'_0 such that $M'_0 \leq_{\mathfrak{K}} N'_0 \in K_\lambda$ and there is a $\leq_{\mathfrak{K}}$-embedding of N_0 into N'_0 over M_0. Now use "(M'_0, M'_1, a) has the extension property" for N'_0. $\blacksquare_{2.11}$

Now we introduce

2.12 Definition: For any models $M, M_0 \in K_\lambda$, any type $p \in \mathcal{S}(M_0)$ and $f_0 \colon M_0 \xrightarrow[\text{iso}]{\text{onto}} M$ we let $\mathcal{S}_p(M) = \mathcal{S}^p_M = \{f_0(f(p)) : f \in \mathrm{AUT}(M_0)\}$. Note: $\mathcal{S}^p_M$ does not depend on f_0. If K is categorical in $\lambda, \mathcal{S}_p(M)$ is well defined for every $M \in K_\lambda$. We write also $\mathcal{S}_{\mathrm{tp}(a,M_0,M_1)}(M)$ or $\mathcal{S}_{(M_0,M_1,a)}(M)$ when $(M_0, M_1, a) \in K^3_\lambda$.

2.13 Claim: *Assume* $(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset$*. If* $(M_0, M_1, a) \in K^3_\lambda$ *is minimal* then *it has the extension property.*

Remark: Instead of $2^{\lambda^+} < 2^{\lambda^{++}}$, we can just demand the definable weak diamond.

Proof: Assume not. By the previous two claims (2.9(1), 2.11) we may assume that (M_0, M_1, a) is minimal. As $\mathfrak{K}$ has amalgamation in λ^+ by 2.2(1), there is $M^* \in K_{\lambda^{++}}$, which is saturated above λ^+ (as $K_{\lambda^{+3}} = \emptyset$), hence M^* is saturated above λ (by 1.10(3)). By 2.6(1) + 2.11, without loss of generality

$\bigotimes_0$ (M_0, M_1, a) is reduced.

Let h: $M_0 \to M^*$ be a $\leq_{\mathfrak{K}}$-embedding and let $p = \text{tp}(a, M_0, M_1)$. If $h(p)$ is realized in M^* by $\geq \lambda^+$ elements we are done by 2.9(2). So assume

$\bigotimes_1$ $h(p)$ is realized by $\leq \lambda$ members of M^*.

Similarly

$\bigotimes_1^+$ q is realized by $\leq \lambda$ members of M^* for $q = g(\text{tp}(a, M_0', M_1'))$ if g is a $\leq_{\mathfrak{K}}$-embedding of M_0' into M^* and $(M_0', M_1', a) \geq (M_0, M_1, a)$.

Next we prove

$\bigotimes_2$ for some reduced $(M_0', M_1', a) \geq (M_0, M_1, a)$ from K_λ^3 we have $|\mathcal{S}_{(M_0', M_1', a)}(M_0')| > \lambda^+$.

Proof of $\bigotimes_2$: If not, we build two non-isomorphic members of K_{λ^+} as follows. First: Choose by induction on $i < \lambda^+, (N_{0,i}, N_{1,i}, a) \in K_\lambda^3$ reduced (see 2.6(1)), increasing continuously (see 2.6(2)), with $N_{0,i} \neq N_{0,i+1}, (N_{0,0}, N_{1,0}, a) = (M_0, M_1, a)$; this is possible as $(N_{0,i}, N_{1,i}, a) \in K_\lambda^3$ has the weak extension property (by 2.4 see 2.3(1)). Let $N^1 = \bigcup_{i<\lambda^+} N_{0,i}$.
Second: Choose by induction on $i < \lambda^+, N_i^0 \leq_{\mathfrak{K}} M^*, \|N_i^0\| = \lambda, N_i^0$ strictly increasing continuous such that:

(*) for every $\beta < \lambda^+, i < \lambda^+$, and $q \in \mathcal{S}_{(N_{0,i}, N_{1,i}, a)}(N_\beta^0)$ for some $\gamma \in (\beta, \lambda^+)$ there are no $N', N_\gamma^0 \leq_{\mathfrak{K}} N' \in K_\lambda$ and $c \in N' \setminus N_\gamma^0$ such that c realize q.

This is straightforward by $\bigotimes_1 + \bigotimes_2$ and bookkeeping. Let $N^0 = \bigcup_{i<\lambda^+} N_i^0$. By categoricity of K_{λ^+} there is an isomorphism g from N^1 onto N^0, so $E = \{\delta < \lambda^+ : g \text{ maps } N_{0,\delta} \text{ onto } N_\delta^0\}$ is a club of λ^+. Now let $\delta^* \in E$, and apply (*) for $\beta = \delta^*, q = g(\text{tp}(a, N_{0,\delta^*}, N_{1,\delta^*}))$ to get γ. Choose $\delta \in E$ which is $> \gamma$. Now $N_{1,\delta}$ gives a contradiction. $\blacksquare_{\bigotimes_2}$

Without loss of generality

$\bigotimes_3$ $|\mathcal{S}_{(M_0, M_1, a)}(M_0)| > \lambda^+$ and $p = \text{tp}(a, M_0, M_1)$.

Next we claim

$\bigotimes_4$ If $M \in K_\lambda, M \leq_{\mathfrak{K}} M^*, \Gamma \subseteq \bigcup\{\mathcal{S}_p(M') : M' \leq_{\mathfrak{K}} M, \|M'\| = \lambda\}$, $|\Gamma| \leq \lambda^+$, then

$$\Gamma^* =: \{q \in \mathcal{S}_p(M) : \text{there is } M', M \leq_{\mathfrak{K}} M', \|M'\| = \lambda,$$
$$M' \text{ realizes } q \text{ but no } c \in M' \backslash M \text{ realizes any } r \in \Gamma\}$$

has cardinality λ^{++}, in fact $|\mathcal{S}_p(M) \backslash \Gamma^*| \leq \lambda^+$.

Proof of $\bigotimes_4$: Without loss of generality $|M^*| = \lambda^{++}$ (i.e. the universe of M^* is λ^{++}). For every $q \in \mathcal{S}_p(M)$ there is a triple $(M_0, M_{1,q}, a_q)$ isomorphic to (M_0, M_1, a) (hence reduced) such that $\text{tp}(a_q, M_0, M_{1,q}) = q$. As M^* is saturated above λ, by 0.26 without loss of generality $M_{1,q} \leq_{\mathfrak{K}} M^*$.

Without loss of generality $\delta < \lambda^{++}$ & (λ^+ divides δ) $\Rightarrow M_\delta =: M^* \restriction \delta \leq_{\mathfrak{K}} M^*$. Now

$(*)_0$ $q_1 \neq q_2 \Rightarrow a_{q_1} \neq q_{q_2}$ and

$(*)_1$ $a_q \notin \delta$ & $\delta < \lambda^{++}$ & λ^+ divides $\delta \Rightarrow M_{1,q} \cap M_\delta = M$.

[Why? As $(M, M_{1,q}, a_q)$ is reduced.]

Now if $r \in \Gamma$, say $r \in \mathcal{S}_p(M'')$, then by $\bigotimes_1$ we know $A_r = \{c \in M^* : c \text{ realizes } r\}$ has cardinality $\leq \lambda$ and hence $A = \bigcup\{A_r : r \in \Gamma\}$ has cardinality $\leq \lambda^+$, so we can find $\delta < \lambda^{++}$ divisible by λ^+ such that $A \subseteq \delta$. But (by $\bigotimes_3$) we have $|\mathcal{S}_p(M)| > \lambda^+$, hence we can find $q[\delta] \in \mathcal{S}_p(M)$ such that $a_{q(\delta)} \notin \delta$, hence $(M, M_{1,q[\delta]}, a_{q[\delta]})$, exemplifies the conclusion of $\bigotimes_4$. $\blacksquare_{\bigotimes_4}$

<u>Final contradiction</u>: By $\bigotimes_4$ we can construct 2^{λ^+} non-isomorphic members of K_{λ^+} using 1.6(1) as follows. We choose by induction on $\alpha < \lambda^+$, for every $\eta \in {}^\alpha 2$, the model M_η and the types p^0_η, p^1_η such that:

(a) $M_{<>} = M_0$,

(b) $M_\eta \in K_\lambda$,

(c) $\langle M_{\eta \restriction \beta} : \beta \leq \ell g(\eta)\rangle$ is $<_{\mathfrak{K}}$-increasing continuous,

(d) $p_\eta \in \mathcal{S}_p(M_{\eta \restriction \beta})$,

(e) for $\beta \leq \alpha$, we have $p^0_\eta, p^1_\eta \in \mathcal{S}(M_\eta)$ and: M_η realizes $p^\ell_{\eta \restriction \beta}$ iff $\beta < \alpha$ & $\ell = \eta(\beta)$.

If $\alpha = 0$ or α is a limit, there is no problem to define M_η for $\eta \in {}^\alpha 2$. If M_η is defined, we can choose, by induction on $i < \lambda^{++}, (N_{\eta,i}, a_{\eta,i})$ such that $(M_\eta, N_{\eta,i}, a_{\eta,i}) \in K^3_\lambda$, $\text{tp}(a_{\eta,i}, M_\eta, N_{\eta,i}) \in \mathcal{S}_p(M_\eta)$ and $N_{\eta,i}$ omits any $q \in \{p^\ell_{\eta \restriction \beta} : \beta < \ell g(\eta), \ell \neq \eta(\beta)\} \cup \{\text{tp}(b, M_\eta, N_{\eta,j}) : j < i \text{ and } b \in N_{\eta,j} \text{ and } \text{tp}(b, M_\eta, N_{\eta,j}) \in \mathcal{S}_p(M_\eta)\}$. By $\bigotimes_4$ we can choose $(N_{\eta,j}, a_{\eta,j})$.

Hence $|W_{\eta,i}| \leq \lambda$, where $W_{\eta,i} = \{j < \lambda^{++} : \text{for some } b \in N_{\eta,i} \text{ we have } \text{tp}(b, M_\eta, N_{\eta,i}) = \text{tp}(a_{\eta,j}, M_\eta, N_{\eta,j})\}$.

Hence we can find $i < j < \lambda^{++}$ such that $i \notin W_{\eta,j}$ & $j \notin W_{\eta,i}$. Let $M_{\eta\hat{}\langle 0\rangle} = N_{\eta,i}, M_{\eta\hat{}\langle 1\rangle} = N_{\eta,j}, p^0_\eta = \operatorname{tp}(a_{\eta,i}, M_\eta, N_{\eta,i}), p^1_\eta = \operatorname{tp}(a_{\eta,j}, M_\eta, N_{\eta,j})$.

Let, for $\eta \in {}^{\lambda^+}2$, $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$, and apply 1.6(1). $\blacksquare_{2.12}$

2.14 Conclusion: $[(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset]$. Every $(M_0, M_1, a) \in K^3_\lambda$ has the extension property.

Proof: By 2.11 and 2.9 + 2.12. $\blacksquare_{2.14}$

2.15 Remark: Conclusion 2.14 says in other words: <u>if</u>

(a) $LS(\mathfrak{K}) \leq \lambda$,

(b) K is categorical in λ and in λ^+,

(c) $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}$,

(d) $K_{\lambda^{+3}}$ is empty,

(e) $2^{\lambda^+} < 2^{\lambda^{++}}$ (or just definable weak diamond),

(f) $\mathfrak{K}$ has amalgamation in λ,

<u>then</u> every triple (M_0, M_1, a) in K^3_λ has the extension property.

2.16 Claim: $[(*)^3_\lambda$, *in other words* $LS(\mathfrak{K}) \leq \lambda; \mathfrak{K}$ *categorical in* λ *and in* λ^+; *and* $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}]$.

If $M_0 \leq_{\mathfrak{K}} M_1$ *are in* K_λ <u>*then*</u> *we can find* $\alpha < \lambda^+$ *and* $\langle N_i : i \leq \alpha\rangle$ *which is* $\leq_{\mathfrak{K}}$*-increasing continuous,* $N_i \in K_\lambda, (N_i, N_{i+1}, a_i) \in K^3_\lambda$ *is reduced,* $M_0 = N_0$, *and* $M_1 \leq_{\mathfrak{K}} N_\alpha$.

Proof: If not, we can contradict categoricity in K_{λ^+} (similar to the proof of $\otimes_2$ during the proof of 2.12).

Without loss of generality $M_0 \neq M_1$. We choose, by induction on $i < \lambda^+$, $N^0_i \in K_\lambda$, $\leq_{\mathfrak{K}}$-increasing continuous such that $(N^0_i, N^0_{i+1}) \cong (M_0, M_1)$ (possible by 2.6(9) and the categoricity of K in λ). Let $N^0 = \bigcup_{i<\lambda^+} N^0_i$.

We choose, by induction on $i < \lambda^+, N^1_i \in K_\lambda$, $\leq_{\mathfrak{K}}$-increasing continuous and a_i such that $(N^1_i, N^1_{i+1}, a_i) \in K^3_\lambda$ is reduced and let $N^1 = \bigcup_{i<\lambda^+} N^1_i$ (possible by 2.6(1) and the categoricity of K in λ). So by the categoricity in λ^+ without loss of generality $N^1 = N^0$, hence for some $\delta_1 < \delta_2 < \lambda^+$ we have

$$N^0_{\delta_1} = N^1_{\delta_1}, N^0_{\delta_2} = N^1_{\delta_2}.$$

By changing names $(N^0_{\delta_1}, N^0_{\delta_1+1}) = (M_0, M_1)$ and so $\langle N_{\delta_1+i} : i \leq \delta_2 - \delta_1\rangle$ is as required. $\blacksquare_{2.16}$

2.17 Conclusion: $[(*)^{2^-}_\lambda + (*)^2_{\lambda^+} + (*)^3_\lambda + 2^{\lambda^+} < 2^{\lambda^{++}} + K_{\lambda^{+3}} = \emptyset$, i.e. the assumption of 2.14].

K_λ has disjoint amalgamation (M_2, M_1 are in disjoint amalgamation over M_0 in M_3 if $M_0 \leq_{\mathfrak{K}} M_\ell \leq_{\mathfrak{K}} M_3, M_1 \cap M_2 = M_0$).

Proof: By 2.16 and iterated applications of 2.14. $\blacksquare_{2.17}$

3. Non-structure

The first major aim of this section is to prove the density of minimal types using as set theoretic assumptions only $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ from cardinal arithmetic. The second aim is to prepare for a proof of a weak form of uniqueness of amalgamation in $\mathfrak{K}_\lambda$. Our aim is also to explain various methods. The proofs are similar to the ones in [Sh 87b, §6].

The immediate role of this section is to get many models in λ^{++} from the assumption "the minimal triples in K^3_λ are not dense": in 3.25 we get this under some additional assumptions, and in 3.27 we get it using only the additional assumption $I(\lambda, K^{+3}) = 0$, which suffices for our main theorem (this does not suffice for the theorem of [Sh 600], see there on this).

But the section is prepared in a more general fashion, so let us first explain two general results concerning the construction of many models based on repeated "failures of amalgamation" or "nonminimality of types".

In 3.19, we give a construction assuming the ideal of small subsets of λ^+ (that is WDmId(λ^+)) is not λ^{++}-saturated, as exemplified by $\langle S_\alpha : \alpha < \lambda^{++}\rangle$. We build for $\eta \in {}^{(\lambda^{+2})>}2$ models $M_\eta \in K_{\lambda^+}$ such that $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta,\alpha}, |M_\eta| = \lambda \times (1 + \ell g \eta)$ and $\nu \triangleleft \eta \Rightarrow M_\nu \leq_{\mathfrak{K}} M_\eta$. Building $M_{\eta\char`^\langle \ell \rangle}$, manufacture $M_{\eta\char`^\langle \ell \rangle}$ as a limit of models $\langle M_{\eta\char`^\langle \ell \rangle,\alpha} : \alpha < \lambda^+\rangle$, a representation of $M_{\eta\char`^\langle \ell \rangle}$, usually in a way predetermined simply, except when $\alpha \in S_{\ell g(\eta)}$ and $\ell = 1$, and then we consult a weak diamond sequence. This is like 1.6(1), but there we use our understanding of models in K_λ to build many models in K_{λ^+} while here we build models in $K_{\lambda^{++}}$, thus getting $2^{\lambda^{+2}}$ models in λ^{+2}. We even get $2^{(\lambda^{+2})}$ models in $K_{\lambda^{+2}}$ with none $\leq_{\mathfrak{K}}$-embeddable into any other.

A second proof 3.23 is like 1.4(1) in the sense that we get only close to $2^{\lambda^{++}}$ models. It is similar to [Sh 87b, 6.4], and the parallel to [Sh 87b, 6.3] holds here. So we have to find an analog of [Sh 87b, definition 6.5, 6.7]. But there we use fullness on the side (meaning: $M \in K_\lambda$ is full over $N \in K_\lambda$ if $N \leq_{\mathfrak{K}} M$, and $(M, c)_{c \in N}$ is saturated), but we do not have this yet.

We still have not explained the framework of this section. In 3.1–3.5 we present construction frameworks $\mathbf{C}$, which involve sequences of models of length $\leq \lambda$ each of cardinality $< \lambda$ and, in particular, define local and nice $\mathbf{C}$. In our applications here, λ^+ plays the role of λ (and $< \lambda^+$ is specialized to $= \lambda$).

Then in 3.6–3.8 we present examples of such frameworks. Our intention is to use the limit of a sequence $\langle M_\alpha : \alpha < \lambda \rangle$ as an approximation to a model of cardinality λ^+. For this we define in 3.10–3.11 a successor relation (next approximation), modulo a "$< \lambda$-amalgamation choice function"; this is denoted $\bar{M}^1 \leq^{\rm at}_{F_1} \bar{M}^2$. Iterating it we get the quasi-order $\leq_F$ (see 3.13). In 3.14 we define the key coding properties (of an amalgamation choice function F for the framework $\mathbf{C}$). The intention is that these coding properties suffice to build many non-isomorphic models in λ^+. In 3.17 we give the "atomic step" for this construction.

In 3.18 we prove the existence of 2^{λ^+} non-isomorphic models, using the λ-coding property. As we do not have this in some applications we have in mind, we next turn to the weak λ-coding property in 3.19 as well as the weak (local) λ-coding property and corresponding properties of F (all in Definitions 3.20 and 3.22), connect them (3.21), and prove that there are many models in 3.23.

Lastly, 3.25 and 3.27 deal with our concrete case: if the minimal triples in $\mathfrak{K}^3_\lambda$ are not dense, then in most cases failures of amalgamation lead to the λ^+-coding property and hence to many models in cardinality λ^{++}.

Note generally that we mainly axiomatize the construction of models in λ^+, not how we get $\bar{M}', \bar{M} \leq^{\rm at}_{F,a} \bar{M}' \in \mathbf{S}eq_\lambda$, that is coding properties; for the last point, see the examples just cited.

Later, in 6.10, we shall need again to use the machinery from this section, in trying to prove that there are enough cases of disjoint amalgamation in $\mathfrak{K}_\lambda$.

We may want to turn the framework presented here into a more general one. See more in [Sh 603].

3.1 CONTEXT: (1) $\mathfrak{K}$ is an abstract elementary class.
(2) But $=_M$ or $=_\mathfrak{K}$ is just an equivalence relation, i.e. for $M \in K, =^M$ is an equivalence relation on $|M|$, moreover a congruence relation relative to all relations (and function symbols which we ignore) in $\tau(M)$, that is for $R \in \tau(M)$ an n-ary relation, we have

$$\bigwedge_{\ell<n} a_i =^M b_i \Rightarrow \langle a_0, \ldots, a_{n-1} \rangle \in R^M \equiv \langle b_0, \ldots, b_{n-1} \rangle \in R^M.$$

We let $\|M\| = |(M)/=^M|$ and

$$K_{\lambda,\mu} = \{M : |M| \text{ has } \mu \text{ elements and } |M|/=^M \text{ has } \lambda \text{ elements}\}.$$

$K_{<\lambda,<\mu}$ are defined naturally, $K_{<\lambda} = K_{<\lambda,<\lambda}$ etc

(3) Now the meaning of $\leq_{\mathfrak{K}}$ should be clear but $M <_{\mathfrak{K}} N$ means ($M \in K, N \in K$ and) $M \subseteq N$ and $M/=^M \leq_{\mathfrak{K}} N/=^N$ and $\exists a \in N[a/=^N \notin (M/=^N)]$, i.e. $(\exists a \in N)(\forall b \in M)(\neg a =^N b)$.
(4) $K^3_\lambda = \{(M, N, a) : M \leq_{\mathfrak{K}} N$ are from $K_{\lambda,\lambda}$ and $a \in N, (a/=^N) \notin (M/=^N)\}$.
(5) In this context "R is an isomorphism relation from M_1 onto M_2" means that
- (a) $R \subseteq M_1 \times M_2$,
- (b) $a_1 =^{M_1} b_1$ & $a_2 =^{M_2} b_2 \Rightarrow a_1 R a_2 \leftrightarrow b_1 R b_2$,
- (c) $(\forall x \in M_1)(\exists y \in M_2) x R y$,
- (d) $(\forall y \in M_2)(\exists x \in M_1) x R y$,
- (e) if $Q \in \tau(M_1) = \tau(M_2)$ is an n-place relation and $a_\ell R b_\ell$ for $\ell = 0, \dots, n-1$ then $(a_0, \dots, a_{n-1}) \in Q^{M_1} \leftrightarrow (b_0, \dots, b_{n-1}) \in Q^{M_2}$,
- (f) $a_1 R a_2$ & $b_1 R b_2 \Rightarrow (a_1 =^{M_1} b_1 \Leftrightarrow a_2 =^{M_2} b_2)$.

3.2 Explanation: The need of 3.1(2) is just to deal with amalgamations which are not necessarily disjoint. If we use disjoint amalgamation, we can omit 3.1(2) below in Definition 3.10, a disappears so F is four place and use K_λ instead of $K_{\lambda,\lambda}$. This is continued in [Sh 603, 2.17t]. Maybe 3.1 would be better understood after reading 3.10, after clause (c).

3.3 Definition: Let λ be regular uncountable and $\mathfrak{K}$ an abstract elementary class.

A λ-construction framework $\mathbf{C} = (\mathfrak{K}^+, \mathbf{Seq}, \leq^*)$ means (we shall use it below with λ^+ playing the role of λ):
- (a) $\tau^+ = \tau^+(\mathfrak{K}^+)$ is a vocabulary extending τ. $\mathfrak{K}^+$ is an abstract elementary class satisfying axioms I, II, III from 0.6 and $M \leq_{\mathfrak{K}^+} N \Rightarrow M \restriction \tau \leq_{\mathfrak{K}} N \restriction \tau$. Furthermore $\mathfrak{K}^+ = \mathfrak{K}^+_{<\lambda}$. As above, equality (in τ) is just a congruence relation.
- (b) $\mathbf{Seq} = \bigcup_{\alpha \leq \lambda} \mathbf{Seq}_\alpha$ where, for $\alpha \leq \lambda$, $\mathbf{Seq}_\alpha$ is a subset of
$$\{\bar{M} : \bar{M} = \langle M_i : i < \alpha \rangle, M_i \in \mathfrak{K}^+_{<\lambda} \text{ is } \leq_{\mathfrak{K}^+} \text{-increasing continuous}\}.$$
For $\alpha = \lambda$ we require further that $M/=^M$ has cardinality λ, where $M = \bigcup_{i<\lambda} M_i$.
- (c) $\leq^*$ is a three place relation on triples x, y, t written $x \leq^*_t y$ for $x, y \in \mathbf{Seq}$ and t a set of pairwise disjoint closed intervals of $\ell g(x)$.

We require:
- (d) $\mathbf{Seq}$ is closed under isomorphism and initial segments.
- (e) If $\bar{M}^1 \leq^*_t \bar{M}^2$ and $\gamma \in \cup t$ <u>then</u> $M^1_\gamma \leq_{\mathfrak{K}^+} M^2_\gamma$ and hence $M^1_\gamma \restriction \tau \leq_{\mathfrak{K}} M^2_\gamma \restriction \tau$ and $\gamma < \ell g(\bar{M}^2)$.
- (f) If $\bar{M}^1 \leq^*_t \bar{M}^2, s \subseteq t$, and $\bar{M}^2 \trianglelefteq \bar{M}^3 \in \mathbf{Seq}$ <u>then</u> $\bar{M}^1 \leq^*_s \bar{M}^3$.

(g) If t is a set of closed pairwise disjoint intervals of $\ell g(\bar{M})$ and $\bar{M} \in \mathbf{Seq}$ then $\bar{M} \leq^*_t \bar{M}$.

3.4 Convention/Definition:

(1) From now on $\mathbf{C}$ will be a λ-construction framework.

(2) If $\bar{M} \in \mathbf{Seq}_\lambda$ then we let $\bar{M} = \langle M_i : i < \lambda \rangle$ and $M =: \bigcup_{i<\lambda} M_i$; similarly with $\bar{M}^x = \langle M^x_i : i < \lambda \rangle$.

(3) $K^{\text{qr}}_\lambda = \{(\bar{M}, \mathbf{f}) : \bar{M} \in \mathbf{Seq}_\lambda \text{ and } \mathbf{f} : \lambda \to \lambda\}$.

(4) If $(\bar{M}^\ell, \mathbf{f}^\ell) \in K^{\text{qr}}_\lambda$ for $\ell = 1, 2$ then $(\bar{M}^1, \mathbf{f}^1) \leq (\bar{M}^2, \mathbf{f}^2)$ means that: for some club E of λ, we have

(a) $\delta \in E \Rightarrow \mathbf{f}^1(\delta) \leq \mathbf{f}^2(\delta)$ and

(b) $\bar{M}^1 \leq^*_t \bar{M}^2$ where $t = t_{E,\mathbf{f}^1} = \{[\delta, \delta + \mathbf{f}^1(\delta)] : \delta \in E\}$.

(5) $\mathbf{Seq}^s = \{\bar{M} \in \mathbf{Seq} : \bigcup_i |M_i| \text{ is a set of ordinals } < \lambda^+\}$; similarly for $\mathbf{Seq}^s_\alpha$.

(6) $K^{\text{qs}}_\lambda = \{(\bar{M}, \mathbf{f}) \in K^{\text{qr}}_\lambda : \bar{M} \in \mathbf{Seq}^s_\lambda\}$.

(7) $\mathbf{C}$ is local (respectively, revised local) if the following clauses (a), (b), (c) hold:

(a) $\bar{M} = \langle M_i : i < \alpha \rangle \in \mathbf{Seq}_\alpha$ iff:

(α) $\bar{M}$ is $\leq_{\mathfrak{K}^+}$-increasing continuous in $\mathfrak{K}^+_{<\lambda}$,

(β) $i + 1 < \alpha \Rightarrow \langle M_i, M_{i+1} \rangle \in \mathbf{Seq}_2$,

(γ) if $\alpha = \lambda$ then $|M/=^M| = \lambda$ (recall $M = \bigcup_{i<\lambda} M_i$);

(b) for $\bar{M}^1, \bar{M}^2 \in \mathbf{Seq}$ and t a set of pairwise disjoint closed intervals contained in $\ell g(\bar{M}^1)$ we have:

$$\bar{M}^1 \leq^*_t \bar{M}^2 \text{ iff } [\gamma_1, \gamma_2] \in t \text{ implies}$$

(α) $\gamma \in [\gamma_1, \gamma_2] \Rightarrow M^1_\gamma \leq_{\mathfrak{K}^+} \bar{M}^2_\gamma$,

(β) in the local case: $\gamma \in [\gamma_1, \gamma_2) \Rightarrow \langle M^1_\gamma, M^1_{\gamma+1} \rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, M^2_{\gamma+1} \rangle$; in the revised local case: if $\ell g(\bar{M}^1), \ell g(\bar{M}^2) < \lambda$ then $\gamma \in [\gamma_1, \gamma_2] \Rightarrow \langle M^1_\gamma, M^1 \rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, M^2 \rangle$, and generally for some club E of $\lambda, \gamma \in [\gamma_1, \gamma_2]$ & $\gamma < \delta \in E \Rightarrow \langle M^1_\gamma, \bigcup_{\beta<\delta} M^1_\beta \rangle \leq^*_{\{[0,1]\}} \langle M^2_\gamma, \bigcup_{\beta<\delta} M^2_\beta \rangle$ (and if $\ell g(\bar{M}^\ell) = \alpha_\ell < \delta$ then $\bigcup_{\beta<\delta} M^\ell_\beta$ means $\bigcup_{\beta<\alpha^\ell} M^\ell_\beta$);

(c) if $\langle M^\zeta_0, M^\zeta_1 \rangle \in \mathbf{Seq}_2$ for $\zeta < \zeta^* < \lambda, \langle M^\zeta_0 : \zeta \leq \zeta^* \rangle$ and $\langle M^\zeta_\ell : \zeta \leq \zeta^* \rangle$ are $\leq_{\mathfrak{K}^+}$-increasing continuous for $\ell = 0, 1$, and $\zeta < \zeta^* \Rightarrow \langle M^\zeta_0, M^\zeta_1 \rangle \leq^*_{\{[0,1]\}} \langle M^{\zeta+1}_0, M^{\zeta+1}_1 \rangle$ then $\langle M^0_0, M^0_1 \rangle \leq^*_{\{[0,1]\}} \langle M^{\zeta^*}_0, M^{\zeta^*}_1 \rangle \in Seq_2$.

So intervals $[\alpha, \alpha] \in t$ are essentially irrelevant for the local version; they just require $M^1_\alpha \leq_{\mathfrak{K}^+} M^2_\alpha$. In the revised local version it is natural to add monotonicity for $\leq_{\{[0,1]\}}$.

(8) For $\alpha \leq \lambda$ we say $\mathbf{C}$ is closed for α if:

(α) $\bar{M} = \langle M_i : i < \alpha\rangle \in \mathbf{Seq}$ iff $\beta < \alpha \Rightarrow \bar{M} \restriction (\beta+1) \in \mathbf{Seq}$,

(β) if $\bar{M}^\ell = \langle M_i^\ell : i < \alpha_\ell\rangle \in \mathbf{Seq}$ for $\ell = 1,2$ and $\alpha = \alpha_1 < \alpha_2$, then

$$\bar{M}^1 \leq_t^* \bar{M}^2 \Leftrightarrow \bar{M}^1 \leq_t^* \bar{M}^2 \restriction \alpha.$$

(9) $\mathbf{C}$ is disjoint if: $\bar{M}^1 \leq_t^* \bar{M}^2, [\gamma_1,\gamma_2] \in t, \gamma \in [\gamma_1,\gamma_2)$ implies $M_\gamma^1 = M_{\gamma+1}^1 \cap M_\gamma^2$. $\mathbf{C}$ is truly disjoint if: $\bar{M}^1 \leq_t^* \bar{M}^2, [\gamma_1,\gamma_2] \in t, \gamma \in [\gamma_1,\gamma_2]$ implies $M_\gamma^1 = M^1 \cap M_\gamma^2$.

(10) In $K_\lambda^{\rm qr}$, we say $(\bar{M},\mathbf{f})$ is a m.u.b. (minimal upper bound) of $\langle(\bar{M}^\xi,\mathbf{f}^\xi) : \xi < \delta\rangle$ if

(a) $\xi < \delta \Rightarrow (\bar{M}^\xi,\mathbf{f}^\xi) \leq (\bar{M},\mathbf{f})$ and

(b) for any $(\bar{M},\mathbf{f}')$ satisfying (a), for some club E of λ we have: if $\alpha \in E$ and $j \leq \mathbf{f}(\alpha)$ then $\mathbf{f}(\alpha) \leq \mathbf{f}'(\alpha)$ and $M_{\alpha+j} \leq_{\mathfrak{K}^+} M'_{\alpha+j}$.

When we require an increasing sequence in $K_\lambda^{\rm qr}$ to be continuous we mean that a m.u.b. is used at limits.

(11) We say $\mathbf{C}$ is explicitly local if it is local and

(d) if $\zeta^* < \lambda$ is a limit ordinal, $\langle M_0^\zeta, M_1^\zeta\rangle \in \mathbf{Seq}_2$ for $\zeta \leq \zeta^*$ and for $\ell = 0,1$ the sequence $\langle M_\ell^\zeta : \zeta < \zeta^*\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, $M_\ell^\zeta \leq_{\mathfrak{K}^+} M_\ell^{\zeta^*}$, and $\zeta < \xi \leq \zeta^* \Rightarrow \langle M_0^\zeta, M_1^\zeta\rangle \leq_{\{[0,1]\}}^* \langle M_0^\xi, M_1^\xi\rangle$ then $\langle \bigcup_{\zeta<\zeta^*} M_0^\zeta, \bigcup_{\zeta<\zeta^*} M_1^\zeta\rangle$ is $\leq_{\{[0,1]\}} (M_0^{\zeta^*}, M_1^{\zeta^*})$.

(12) $\mathbf{C}$ is closed if it is closed for every ordinal $\leq \lambda$.

(13) $\mathbf{C}$ is semi (respectively almost) closed, as witnessed by G, if:

(a) $\mathbf{C}$ is closed for every limit ordinal $\delta < \lambda$;

(b) G is a function from $\mathbf{Seq}_{<\lambda}$ to $\mathbf{Seq}_{<\lambda}$ such that $\bar{M} \lhd G(\bar{M})$;

(c) $\bar{M} = \langle M_\alpha : \alpha < \lambda\rangle$ belongs to $\mathbf{Seq}_\lambda$ if $\bar{M}$ obeys G, which means: $\beta < \lambda \Rightarrow \bar{M} \restriction \beta \in \mathbf{Seq}_\beta$ and $\{\alpha < \lambda : G(\bar{M} \restriction \alpha) \lhd \bar{M}\}$ is unbounded in λ;

(d) in the almost closed version, we add: $G(\bar{M})$ depends on $\bigcup \bar{M} = \bigcup_{i<\lg \bar{M}} M_i$ only.

(14) $\mathbf{C}$ is λ-nice if

(a) $\leq$ is a transitive on K_λ^{qr};

(b) any increasing continuous sequence in K_λ^{qr} of length $< \lambda^+$ has a m.u.b. (see part (10)) (not necessarily unique);

(c) $\mathbf{C}$ is closed (see part (12)).

(15) $\mathbf{C}$ is almost λ-nice (as witnessed by G) is defined similarly, replacing "closed" by "almost closed" (witnessed by G).

3.5 Claim: *Let $\mathbf{C}$ be a local (or revised local) λ-construction framework.*

(1) *If $(\bar{M}^\ell,\mathbf{f}^\ell) \in K_\lambda^{\rm qr}$ for $\ell = 1,2$ and E is a club of λ^+ and $(*)$ below then $(\bar{M}^1,\mathbf{f}^1) \leq (\bar{M}^2,\mathbf{f}^2)$ when*

(*) *if* $\delta \in E$ *then* $M^1_{\delta+i} = M^2_{\delta+i}$ *for* $i \leq \mathbf{f}^1(\delta)$ *and* $\mathbf{f}^1(\delta) \leq \mathbf{f}^2(\delta)$*; in the "revised local" version assume in addition that* $M^1 = M^2$.

(2) $\leq$ *is a transitive and reflexive relation on* K^{qr}_λ.

(3) *Any increasing continuous sequence of pairs from* K^{qr}_λ *of length* $< \lambda^+$ *has a minimal upper bound.*

(4) *If, in addition,* **C** *is explicitly local (see Definition* 3.4(11)*)* *then* *any increasing sequence in* K^{qr}_λ *of length* $< \lambda^+$ *has a lub.*

(5) **C** *is* λ*-nice (hence, in particular,* λ*-closed).*

Proof: (1) Check clause (b) of Definition 3.4(7) and Definition 3.4(4).

(2) Use clauses (b), (c) of Definition 3.4(7)). In (c) take $\zeta^* = 2$.

(3) Without loss of generality the elements of the sequence are $(\bar{M}^\xi, \mathbf{f}^\xi) \in K^{\mathrm{qr}}_\lambda$ for $\xi < \mu$, where μ is a regular cardinal $\leq \lambda$. For $\xi < \zeta < \mu$, let $E_{\xi,\zeta}$ be a closed unbounded subset of λ exemplifying Definition 3.4(4) for $(\bar{M}^\xi, \mathbf{f}^\xi) \leq (\bar{M}^\zeta, \mathbf{f}^\zeta)$. First, when $\mu < \lambda$, let $E \subseteq \bigcap_{\xi<\zeta<\mu} E_{\xi,\zeta} \subset \lambda$ be a closed unbounded subset of λ such that $\alpha \in E \Rightarrow \alpha + (\sup_{\xi<\mu} \mathbf{f}^\xi(\alpha)) + 1 < \mathrm{Min}(E \backslash (\alpha+1))$. Second, when $\mu = \lambda$ let $E \subseteq \{\delta < \lambda : \delta \in \bigcap_{\xi<\zeta<\delta} E_{\xi,\zeta}\} \subseteq \lambda$ be a closed unbounded subset of λ such that $\alpha \in E \Rightarrow \alpha + \sup_{\xi<\alpha} \mathbf{f}^\xi(\alpha) + 1 < \mathrm{Min}(E\backslash(\alpha+1))$. We concentrate on the first case for notational simplicity. Let $E = \{\alpha_i : i < \lambda\}$ with α_i increasing continuous with i. Notice that for every i, $\xi < \zeta < \mu \Rightarrow \mathbf{f}^\xi(\alpha_i) \leqq \mathbf{f}^\zeta(\alpha_i)$. Let $E^* = \{i : \alpha_i = i\}$. We now define $\bar{M} = \langle M_j : j < \lambda \rangle$ by defining M_j by induction on j. If $j = \alpha_j \in E^*$ let $M_j = \bigcup_{\xi<\mu} M^\xi_j$. If $\alpha < j \leq \alpha + \mathbf{f}^\xi(\alpha)$ for some $\alpha \in E^*$ and some $\xi < \mu$, let $\xi' = \sup\{\xi : \alpha + \mathbf{f}^\xi(\alpha) \leq j\}$ and set $M_j = \bigcup_{\xi>\xi'} M^\xi_j$. If $j = \sup_\xi(\alpha + \mathbf{f}^\xi(\alpha))$ for some $\alpha \in E$ and $j > \alpha + \mathbf{f}^\xi(\alpha)$ for each $\xi < \mu$, let $M_j = \bigcup_{\beta<j} M_\beta$. Finally, if $j < \lambda$ does not fall under any of the previous cases, let $M_j = \bigcup_{\xi<\mu} M^\xi_{\alpha_j}$.

We claim that $\bar{M} \in \mathbf{Seq}_\lambda$. One checks that $\bar{M}$ is continuous and increasing, the main point being that if $\alpha \in E^*$ and $\alpha < j_1 < \alpha + \mathbf{f}^{\xi_1}(\alpha) \leq j_2 < \alpha + \mathbf{f}^{\xi_2}(\alpha)$ for $\xi_1 < \xi_2 < \mu$, then $M^{\xi_1}_{j_1} \leq M^{\xi_2}_{j_1} \leq M^{\xi_2}_{j_2}$. One must also check that $\langle M_j, M_{j+1} \rangle \in \mathbf{S}eq_2$ for all j. This follows from clause (c) of Definition 3.4(7).

Let $\mathbf{f}$ be defined by $\mathbf{f}(\alpha_i) = \sup\{\mathbf{f}^\xi(\alpha_i) : \xi < \mu\}$ if $i \in E^*$ and $\mathbf{f}(\alpha_i) = 0$ otherwise. Clearly $(\bar{M}^\xi, \mathbf{f}^\xi) \leq (\bar{M}, \mathbf{f})$ for $\xi < \mu$.

What about being a $\leq$-m.u.b.? Assume that $(\bar{M}', \mathbf{f}') \in K^{qr}_\lambda$ and $\xi < \mu \Rightarrow (\bar{M}^\xi, \mathbf{f}^\xi) \leq (\bar{M}', \mathbf{f}')$. So for each $\xi < \mu$ some club E'_ξ of λ exemplifies Definition 3.4(4), and let $E' =: \bigcap_{\xi<\mu} E'_\xi \cap E^*$, a club of λ.

Now for $\delta \in E'$ we have $(\forall \xi < \mu)(\mathbf{f}^\xi(\delta) \leq \mathbf{f}'(\delta))$, hence $\mathbf{f}(\delta) = \sup_{\xi<\mu} \mathbf{f}^\xi(\delta) \leq \mathbf{f}'(\delta)$, so $\delta \in E' \Rightarrow \mathbf{f}(\delta) \leq \mathbf{f}'(\delta)$. Similarly $\delta \in E'$ & $j \leq \mathbf{f}(i) \Rightarrow M_{\delta+j} \leq_{\mathfrak{K}} M'_{\delta+j}$. So clearly $(\bar{M}, \mathbf{f}) \leq (\bar{M}', \mathbf{f}')$ and $(\bar{M}, \mathbf{f})$ is a minimal u.b. (see Definition 3.4(10)!).

(4) As in the proof of part (3), let $\langle(\bar{M}^\xi, \mathbf{f}^\xi) : \xi < \mu\rangle$ be as therein and let $(\bar{M}, \mathbf{f})$ be constructed as above. For proving it is a lub, let $(M^\xi, \mathbf{f}^\xi) \leq (\bar{M}', \mathbf{f}')$ for $\xi < \mu$, and define E' as there. For $\delta \in E', j < \mathbf{f}(\delta)$ we have $\langle(M^\xi_{\delta+j}, M^\xi_{\delta+j+1}) : \xi \in (\xi_{\delta,j+1}, \mu)\rangle$ is $\leq^*_{\{[0,1]\}}$-increasing continuous and $\langle M^\xi_{\delta+j}, M^\xi_{\delta+j+1}\rangle \leq^*_{\{[0,1]\}} (M'_{\delta+j}, M'_{\delta+j+1})$ for $\xi \in (\xi_{\delta,j+1}, \mu)$, so as $\mathbf{C}$ is explicitly local by clause (d) in Definition 3.4(11) we have

$$\langle M_{\delta+j}, M_{\delta+j+1}\rangle = \left\langle \bigcup_{\xi\in(\xi_{\delta,j+1},\mu)} M^\xi_{\delta+j}, \bigcup_{\xi\in(\xi_{\delta,j+1},\mu)} M^\xi_{\delta+j+1} \right\rangle \leq_{\{[0,1]\}} (M'_{\delta+j}, M'_{\delta+j+1}),$$

as required.

The proof for the case $\mu = \lambda$ is similar, using diagonal intersection.

(5) Left to the reader. $\blacksquare_{3.5}$

* * *

It may clarify matters if we introduce some natural cases of $\mathbf{C}$. We shall use the forthcoming $\mathbf{C}^0$ in our construction of many models in $\mathfrak{K}_{\lambda^{+2}}$.

3.6 Definition: For $\ell \in \{0,1,2\}$ and $\lambda = \mathrm{cf}(\lambda) > LS(\mathfrak{K})$, let $\mathbf{C} = \mathbf{C}^\ell_{\mathfrak{K},\lambda}$ consist of

(a) $\tau^+ = \tau$, $\mathfrak{K}^+ = \{M \in \mathfrak{K}_{<\lambda} :$ if λ is a successor cardinal then $\|M\|^+ = \lambda\}$ (with $=^M$ being equality),

(b) $\mathbf{Seq}_\alpha = \{\bar{M} : \bar{M} = \langle M_i : i < \alpha\rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of $\mathfrak{K}_{<\lambda}$ and if $\alpha = \lambda$ then $\bigcup_{i<\alpha} M_i$ has cardinality $\lambda\}$,

(c) $\bar{M} <^*_t \bar{N}$ when:
  - (α) $\bar{M} = \langle M_i : i < \alpha^*\rangle, \bar{N} = \langle N_i : i < \beta^*\rangle$ are from $\mathbf{Seq}$,
  - (β) if $[\gamma_1, \gamma_2] \in t$ then:
    - (i) $\gamma \in [\gamma_1, \gamma_2] \Rightarrow M_\gamma \leq_{\mathfrak{K}} N_\gamma$,
    - (ii) if $\ell = 1$, then in addition $\gamma \in [\gamma_1, \gamma_2) \Rightarrow M_\gamma = M_{\gamma+1} \cap N_\gamma$,
    - (iii) if $\ell = 2$, then in addition $\gamma \in [\gamma_1, \gamma_2) \Rightarrow M_\gamma = M \cap N_\gamma$ where $M = \bigcup_{i<\alpha^*} M_i$.

Note that $\mathbf{C}^1_{\mathfrak{K},\lambda}, C^2_{\mathfrak{K},\lambda}$ are interesting when we have disjoint amalgamation in the appropriate cases.

3.7 FACT:
(1) If $\ell = 0$ or 1, then $\mathbf{C}^{\ell}_{\mathfrak{K},\lambda}$ is an explicitly local λ-construction framework (hence λ-nice by 3.5(5)) and $\mathfrak{K}^+$ satisfies axioms I–VI.
(2) If $\ell = 0$ or 2, then $\mathbf{C}^{\ell}_{\mathfrak{K},\lambda}$ is an explicitly revised local λ-construction framework (hence λ-nice by 3.5(5)) and $\mathfrak{K}^+$ satisfies axioms I–VI.

3.8 Definition: If $\lambda =$ cf$(\lambda) > LS(\mathfrak{K})$ then $\mathbf{C}^{3}_{\mathfrak{K},\lambda}$ consists of:

(a) $\tau^+ = \tau \cup \{P, <\}, \mathfrak{K}^+$ is the set of $(M, P^M, <^M)$ where $M \in \mathfrak{K}_{<\lambda}, P^M \subseteq M, <^M$ a linear ordering of P^M (but $=^M$ may be as in 3.1(2)) and $M_1 \leq_{\mathfrak{K}^+} M_2$ iff $(M_1 \restriction \tau) \leq_{\mathfrak{K}} (M_2 \restriction \tau)$ and $M_1 \subseteq M_2$,

(b) $\mathbf{Seq}_\alpha = \{\bar{M} : \bar{M} = \langle M_i : i \leq \alpha\rangle$ is an increasing continuous sequence of members of $\mathfrak{K}^+$ and $\langle M_i \restriction \tau : i \leq \alpha\rangle$ is $\leq_{\mathfrak{K}}$-increasing, and for $i < j < \alpha$: P^{M_i} is a proper initial segment of $(P^{M_j}, <^{M_j})$ and there is a first element in the difference$\}$, we denote the $<^{M_{i+1}}$-first element of $P^{M_{i+1}} \backslash P^{M_i}$ by $a_i[\bar{M}]$,

(d) $\bar{M} <^*_t \bar{N}$ iff $\bar{M} = \langle M_i : i < \alpha^*\rangle, \bar{N} = \langle N_i : i < \alpha^{**}\rangle$ are from $\mathbf{Seq}, t$ is a set of pairwise disjoint closed intervals of α^* and for any $[\alpha, \beta] \in t$ we have ($\beta < \alpha^*$ and): $\gamma \in [\alpha, \beta) \Rightarrow M_\gamma \leq_{\mathfrak{K}} N_\gamma$ & $a_\gamma[\bar{M}] \notin N_\gamma$, moreover $a_\gamma[\bar{M}] = a_\gamma[\bar{N}]$.

3.9 FACT: $\mathbf{C}^{3}_{\mathfrak{K},\lambda}$ is an explicitly local and revised local λ-construction framework (hence λ-nice by 3.5(5)) and $\mathfrak{K}^+$ satisfies axioms I–VI.

We now introduce amalgamation choice functions. The use of "F a λ-amalgamation choice function" is to help use the weak diamond, by taking out most of the freedom in choosing amalgams. This gives possibilities for coding (3.14, 3.17).

3.10 Definition: (1) We say that F is a λ-amalgamation choice function for the construction framework $\mathbf{C}$ iff F is a five place function satisfying:

(a) if $M_\ell \in K^+_{<\lambda}$ for $\ell < 3, M_0 \leq_{\mathfrak{K}^+} M_1, M_0 \leq_{\mathfrak{K}^+} M_2, M_1 \cap M_2 = M_0$ (before dividing by $=^{M_\ell}$); $a \in M_2$ and $(\forall b \in M_0)[\neg a =^{M_1} b]$; and A is a set such that $A \cup |M_1| \cup |M_2|$ is a set of ordinals then $F(M_0, M_1, M_2, A, a)$, if defined, is a member N of $\mathfrak{K}^+$ with universe $A \cup |M_1| \cup |M_2|$, which $\leq_{\mathfrak{K}^+}$-extends M_1 and M_2 and $(a/=^N) \notin (M_1/=^N)$;

(b) [uniqueness] if (M_0, M_1, M_2, A, a) and $(M'_0, M'_1, M'_2, A', a')$ are as above and in the domain of F, f is an order preserving mapping from $A \cup |M_1| \cup |M_2|$ onto $A' \cup |M'_1| \cup |M'_2|$ and, for technical reasons, $(\forall \alpha, \beta \in \mathrm{Dom}(f)[\alpha, \beta < \lambda^+ \to \alpha = \beta + 1 \equiv f(\alpha) f(\beta) + 1]$ (see 3.21(2)'s proof) such that $f \restriction M_\ell$ is an isomorphism from M_ℓ onto M'_ℓ for $\ell = 0, 1, 2$ (so preserving $=^{M_\ell}$, and its negation) and $f(a) = a'$ then f is an isomorphism from $F(M_0, M_1, M_2, A, a)$ onto $F(M'_0, M'_1, M'_2, A', a')$;

(c) if $F(x_0, x_1, x_2, x_3, x_4)$ is well defined then x_0, x_1, x_2, x_3, x_4 are as in part (a).

Observe that as $M_1 \cap M_2 = M_0$ in (a), if we do not have disjoint amalgamation then we are *forced* to allow $=_N$ to be a nontrivial congruence.

(2) If F is defined whenever the conditions in part (a) hold and $A \backslash M_1 \backslash M_2$ has large enough cardinality then we say F is full (if $\lambda = \mu^+$, it suffices to demand $A \backslash M_1 \backslash M_2$ has cardinality μ).

(3) We say F has strong uniqueness if

(d) if (M_0, M_1, M_2, A, a) and $(M'_0, M'_1, M'_2, A', a')$ are as above and in the domain of F and for $\ell = 0, 1, 2$ we have R_ℓ is an isomorphism relation from M_ℓ onto M'_ℓ such that $R_0 = R_1 \cap (M_0 \times M'_0) = R_2 \cap (M_0 \times M'_0)$ and $|A \backslash M_1 \backslash M_2| = |A' \backslash M'_1 \backslash M'_2|$, then there is an isomorphism relation R from $M = F(M_0, M_1, M_2, A, a)$ onto $M' = F(M'_0, M'_1, M'_2, A', a)$ such that $R_\ell = R \cap (M_\ell \times M'_\ell)$ for $\ell = 0, 1, 2$.

3.11 Definition: Assume $\mathbf{C}$ is a λ-construction framework and F is a λ-amalgamation choice function for $\mathbf{C}$. Let $(\bar{M}^\ell, \mathbf{f}^\ell) \in K^{\mathrm{qr}}_\lambda$ for $\ell = 1, 2$.

(1) $(\bar{M}^1, \mathbf{f}^1) <^{\mathrm{at}}_{F,a} (\bar{M}^2, \mathbf{f}^2)$ (if we omit a, this means for some a; "at" stands for *atomic* extension; we may write $\leq^{\mathrm{at}}_{F,a}$ instead of $<^{\mathrm{at}}_{F,a}$) means that:

(a) $(\bar{M}^1, \mathbf{f}^1) \leq (\bar{M}^2, \mathbf{f}^2)$,

(b) for some club E of λ, for every $\delta \in E$ taking $e_\delta =: [\delta, \delta + \mathbf{f}^1(\delta)]$ we have

(∗) if $\beta < \gamma$ are successive members of e_δ then:

$$M^2_\gamma = F(M^1_\beta, M^1_\gamma, M^2_\beta, |M^2_\gamma|, a),$$

(∗∗) $|M^2_\gamma| = |M^1_\gamma| \cup |M^2_\beta| \cup \{i \in M^2_\gamma : i$ an ordinal not in $|M^1_\gamma| \cup |M^2_\beta|$ and $\mathrm{otp}(|M^2_\gamma| \cap i \backslash |M^1_\gamma| \backslash |M^2_\beta|) < \|M^2_\gamma\|\}$,

(∗∗∗) $\|M^2_\gamma\| = \mathrm{Min}\{\|N\| : N = F(M^1_\beta, M^1_\delta, M^2_\beta, |N|, a)\}$.

A suitable club E may be called a *witness* for the relation. Implicit in clause (b) is $a \in M^2$ and $\neg(\exists b)(b \in M^1 \ \&\ a =^{M^2} b)$.

(2) $(\bar{M}', \mathbf{f}') \leq_F (\bar{M}'', \mathbf{f}'')$ means that: there is a sequence $\langle (\bar{M}^\zeta, \mathbf{f}^\zeta) : \zeta \leq \xi \rangle$ such that:

(a) $\xi < \lambda^+$,

(b) $(\bar{M}^\zeta, \mathbf{f}^\zeta) \in K^{\mathrm{qr}}_\lambda$ is $\leq$-increasing continuous in ζ (remember Definition 3.4(10)),

(c) for each $\zeta < \xi$ we have $(\bar{M}^\zeta, \mathbf{f}^\zeta) \leq^{\mathrm{at}}_F (\bar{M}^{\zeta+1}, \mathbf{f}^{\zeta+1})$,

(d) $(\bar{M}', \mathbf{f}') = (\bar{M}, \mathbf{f}^0)$ and $(\bar{M}'', \mathbf{f}'') = (\bar{M}^\xi, \mathbf{f}^\xi)$.

A club E which witnesses all the relations in (c), or at least each of them is witnessed by some end segment of E, is called a witness for the relation $\leq_F$.
(3) $<^{\mathrm{at},*}_{F,a}$ is defined similarly to part (1) but we demand in clause (b) only that $e_\delta \subseteq [\delta, \delta + \mathbf{f}^2(\delta)]$ is closed and $\{\delta, \delta + \mathbf{f}^2(\delta)\} \subseteq e_\delta$; the requirement from clause (a) is unchanged and we require also:

$$\text{if } \beta \in [\delta, \delta + \mathbf{f}^2(\delta)] \text{ then } M^2_\beta = M^2_{\max(e_\delta \cap (\beta+1))}.$$

Then define $\leq^*_F$ by iterating $\leq^{at,*}_F$.
(4) We may replace F by $\mathbf{F}$, a family of such functions. Then in each case in 3.11(2)(c) we use one such F. $\mathbf{F}^*$ is the family of all such F's.
(5) $(\bar{M}^1, \mathbf{f}^1) <_{F,a} (\bar{M}^2, \mathbf{f}^2)$ means that for some $(\bar{M}, \mathbf{f})$ we have $(\bar{M}^1, \mathbf{f}^1) \leq^{at}_{F,a} (\bar{M}, \bar{f}) \leq_F (\bar{M}^2, \mathbf{f}^2)$.
(6) We define mub as in 3.4(10).

3.12 Remark: (1) What we prove below on $<^{\mathrm{at}}_{F,a}, \leq_F$ also holds for $<^{\mathrm{at},*}_{F,a}, \leq^*_F$.
(2) Note: using $\mathbf{F}$ rather than F may help in proving cases of Definition 3.20, but we can use one F which codes all members of $\mathbf{F}$ by asking on $A \backslash M_1 \backslash M_2$, though artificially.
(3) We can replace F by $\langle F_\eta : \eta$ a sequence of ordinals of length $< \lambda, \eta(1+i) < 2, \eta(0) < 2^{<\lambda}\rangle$, each F_η with uniqueness 3.10(3) and $(**)$ of 3.11(1)(b) is replaced by $M^2_\gamma = f_{\eta \restriction \delta}(M^1_\beta, M^1_\gamma, M^2_\beta, |M^2_\gamma|, a)$, and omit $(***)$ there.

3.13 CLAIM: *If $\mathbf{C}$ is nice, <u>then</u> on $K^{\mathrm{qr}}_\lambda, \leq_F$ is a quasi-order, and every increasing continuous sequence of length less than λ^+ has a mub.*

Proof: Check. $\blacksquare_{3.13}$

3.14 Definition: (1) We say a λ-amalgamation choice function F for $\mathfrak{K}^+$ has the *λ-coding property* for $\mathbf{C}$ if: $\mathbf{Seq}_\lambda \neq \emptyset$, and for every $\bar{M}^1 \in \mathbf{Seq}_\lambda$, function $\mathbf{f}^1 : \lambda \to \lambda$, and $S \subseteq \lambda$ we can find $\bar{M}^{2,\eta} \in \mathbf{Seq}_\lambda$ for $\eta \in {}^\lambda 2$ with η extending $0_{\lambda \backslash S}$, that is $\eta \restriction (\lambda \backslash S)$ being constantly zero, a function $\mathbf{f}^2 : \lambda \to \lambda$ such that $\mathbf{f}^1 \leq_{\mathcal{D}_\lambda} \mathbf{f}^2, \mathbf{f}^2 \restriction (\lambda \backslash S) = \mathbf{f}^1 \restriction (\lambda \backslash S)$ and an element a_η of $M^{2,\eta}$ (usually $a_\eta = a$) such that:

- $(*)_1$ $(\bar{M}^1, \mathbf{f}^1) \leq^{\mathrm{at}}_{F,a} (\bar{M}^{2,\eta}, \mathbf{f}^1)$ for all η extending $0_{\lambda \backslash S}$, and $\eta \restriction \alpha = \nu \restriction \alpha \Rightarrow M^{2,\eta} \restriction \alpha = M^{2,\nu} \restriction \alpha$; and $(\overline{M}^1, \mathbf{f}^1) \in K^{qs}_\lambda$ implies $(\overline{M}^{2,\eta}, \mathbf{f}^1) \in K^{qs}_\lambda$.
- $(*)_2$ for some club E of λ the following is impossible: for some $\eta_3, \eta_4 \in {}^\lambda 2$ extending $0_{\lambda \backslash S}$, for $\ell = 3,4$ we have $(\bar{M}^1, \mathbf{f}^2) \leq_{F,a^\ell} (\bar{M}^\ell, \mathbf{f}^\ell)$ witnessed by a club E_ℓ, we have (abusing our notation we are dividing by the equality congrugence) f_ℓ a $\leq_{\mathfrak{K}}$-embedding of $M^{2,\eta_\ell} \restriction \tau$ into $M^\ell \restriction \tau$ over M^1_λ, and

for some $\delta \in E \cap E_3 \cap E_4 \cap S$ we have $\bar{M}^3 \restriction \delta = \bar{M}^4 \restriction \delta, a^3 \in M_\delta^{2,\eta_3}, a^4 \in M_\delta^{2,\eta_4}, f_3(a^3) = f_4(a^4), \mathbf{f}^3 \restriction \delta = \mathbf{f}^4 \restriction \delta, f_3 \restriction M_\delta^{2,\eta_3} = f_4 \restriction M_\delta^{2,\eta_4}$, and $\ell \in \{3,4\} \Rightarrow \text{Rang}(f_\ell \restriction M^{2,\eta_\ell}) \subseteq M_\delta^\ell, \eta_3 \restriction \delta = \eta_4 \restriction \delta, \eta_3(\delta) \neq \eta_4(\delta)$.

(2) We say that F has the weak λ-coding property if above we restrict ourselves to the cases $\mathbf{f}^1 \restriction S = 0_S$. We can even restrict ourselves to the cases $\mathbf{f}^1 \in \mathcal{F} \subseteq {}^\lambda\lambda$ provided that $0_\lambda \in \mathcal{F}$ and we demand $\mathbf{f}^2 \in \mathcal{F}$.

(3) If we replace F by $\mathbf{F}$, a family of such functions, it means we use Definition 3.11(4).

(4) We say $\mathfrak{K}$ has a coding property if some λ-amalgamation choice function F has this property. Typically, the actual choice of F is irrelevant as long as its domain is sufficiently rich.

3.15 Observation: The restriction above to η such that η extends $0_{\lambda \setminus S}$ is natural but inessential, as we can extend the definition of $M^{2,\eta}$ to all η in ${}^\lambda 2$ by defining $M^{2,\eta} = M^{2,\eta'}$ where $\eta' \restriction S = \eta \restriction S$ and $\eta' \restriction (\lambda \setminus S)$ is constantly zero. Then the same properties will hold.

3.16 Remarks:

(1) For a local construction framework $\mathbf{C}$ in 3.14(1) the conditions $(*)_1$ and $(*)_2$ can be replaced by local requirements. For example, in condition $(*)_1$, we may take 3.4(7b) into account.

(2) In 3.14$(*)_2$ a sufficient condition for the impossibility of the stated conditions on E, η^3, η^4 and δ, where $\eta^3 \restriction \delta = \eta^4 \restriction \delta = \eta$, say, is that there is $\bar{a} \in M_\delta^{2,\eta}$ so that:

$(*)_3$ $\text{tp}(\bar{a}, M^1_{\delta+\mathbf{f}^2(\delta)}, M^{2,\eta^3}) \neq \text{tp}(\bar{a}, M^1_{\delta+\mathbf{f}^2(\delta)}, M^{2,\eta^4})$.

We can even allow $\bar{a}$ to be infinite here, say a full listing of $M_\delta^{2,\eta}$.

To see that this suffices, suppose that we also have the conditions of 3.14$(*)_2$. Then for $\ell = 3, 4$ as $M^{2,\eta_\ell}_{\delta+j+1}$ is given by

$$F(M^1_{\delta+j}, M^1_{\delta+j+1}, M^{2,\eta_\ell}_{\delta+j}, |M^{2,\eta_\ell}_{\delta+j+1}|, a^\ell)$$

where $j < \mathbf{f}^2(\delta)$, we find $M^{2,\eta_3}_{\delta+\mathbf{f}^2(\delta)} = M^{2,\eta_4}_{\delta+\mathbf{f}^2(\delta)} = M^*$, say.

Since M^{2,η_3} and M^{2,η_4} can be amalgamated over M^*, we have

$$\text{tp}(f_3(\bar{a}), M^1_{\delta+\mathbf{f}^2(\delta)}, M^{3,\eta_3}) = \text{tp}(f_4(\bar{a}), M^1_{\delta+\mathbf{f}^2(\delta)}, M^{4,\eta_4}).$$

On the other hand, by $(*)_2$ we have

$$\text{tp}(\bar{a}, M^1_{\delta+\mathbf{f}^2(\delta)}, M^{2,\eta_\ell}) = \text{tp}(f_\ell(\bar{a}), M^1_{\delta+\mathbf{f}^2(\delta)}, M^{\ell,\eta_\ell})$$

and this gives a contradiction.

(3) To understand Definitions 3.14(1,2) you may look at the places they are verified, such as 3.25 and 3.27. Also see 3.21(3,4).

(4) The next lemma deduces from the criterion in 3.14(1) another one which is natural for use in a non-structure theorem.

(5) Note that 3.14(1) implies: for every $(\bar{M}^1, \mathbf{f}^1)$ there is $(\bar{M}^2, \mathbf{f}^2)$ such that $(\bar{M}^1, \mathbf{f}^1) \leq^{at}_{F,a} (\bar{M}^2, \mathbf{f}^2)$ and $M^1_\lambda \neq M^2_\lambda$ and even $M^1_\lambda / =^{M^2_\lambda}$, $M^2_\lambda / =^{M^2_\lambda}$ are not equal.

(6) In Definition 3.14(1) we can replace $<^{at}_{F,a}$ by $<_F$ or $<_{F,a}$ with no harm as $<_{F,a}$ satisfies the requirement on $<^{at}_{F,a}$ and starting from it we again get $<_F$.

(7) In $(*)_2$ of Definition 3.14(1) for some function H depending on $(\bar{M}^1, f^1)$ we may add the further restriction: $\ell \in \{3,4\}$ and $\alpha < \delta$ implies $\delta \in H(\bar{M}^\ell \restriction \alpha, f_\ell \restriction M^\ell_\alpha)$ when the latter is a club of λ; i.e. this weakening of the demand does not change the desired conclusions.

(8) We can weaken the demand in $(*)_2$ of 3.14(1) to extensions which actually arise but this seems more cumbersome. While the adaptation is straightforward, we have no application in mind.

(9) In 3.14(1), $(*)_2$ we may strengthen the requirement by excluding the case where the club E is allowed to depend on η. That is, we consider quadruples $(E^\eta, \bar{M}^\eta, \mathbf{f}^\eta, f^\eta)$ for $\eta \in {}^\lambda 2$ such that $(\bar{M}^1, \mathbf{f}^2) \leq_F (\bar{M}^\eta, \mathbf{f}^\eta)$ is witnessed by a club E^η in λ and f^η is a $\leq_{\mathfrak{K}}$-embedding of $M^{2,\eta} \restriction \tau$ into $M^\eta \restriction \tau$ over M^1. We require:

$(*)'_2$ for no $\eta^3, \eta^4 \in {}^\lambda 2$ and $\delta \in E^{\eta^3} \cap E^{\eta^4} \cap E \cap S$ do we have:

$$\eta_3 \restriction \delta = \eta_4 \restriction \delta, \quad \eta_3(\delta) \neq \eta_4(\delta), \quad \mathbf{f}^3 \restriction \delta = \mathbf{f}^4 \restriction \delta;$$
$$\bar{M}^{2,\eta_3} \restriction [\delta, \delta + \mathbf{f}^2(\delta)] = \bar{M}^{2,\eta_4} \restriction [\delta, \delta + \mathbf{f}^3(\delta)];$$
$$f^{\eta_3} \restriction M^{2,\eta_3}_\delta = f^{\eta_4} \restriction M^{2,\eta_4}_\delta;$$
$$f^{\eta_\ell}[M^{2,\eta_\ell}_\delta] \subseteq M^{\eta_\ell}_\delta \quad \text{for } \ell = 3,3.$$

(10) In 3.14 we can also require the models $M^1, M^{2,\eta}$ to have universes $\lambda(1+\alpha)$ and $\lambda(1+\alpha+1)$ respectively for some α, with $\lambda(1+\alpha) \in M^{2,\eta}_0$. This will not change much.

3.17 LEMMA: *Assume* $(\exists\mu)(\lambda = \mu^+ \ \& \ 2^\mu < 2^{\mu^+})$ *or at least the definitional weak diamond on* λ *holds. Assume* $\mathbf{C}$ *is* λ*-nice,* $J = \mathrm{WDmId}^{\mathrm{Def}}(\lambda)$.

If the λ*-amalgamation choice function* F *has the* λ*-coding property, then it has the explicit* (λ, J)*-coding property, which means: if* $(\bar{M}^1, \bar{\mathbf{f}}^1) \in K^{\mathrm{qs}}_\lambda$ *and* $S \subseteq \lambda$ *satisfies* $S \notin J$ *then we can find* $(\bar{M}^2, \mathbf{f}^2) \in K^{\mathrm{qs}}_\lambda$ *such that:*

(a) $(\bar{M}^1, \mathbf{f}^1) \leq^{\mathrm{at}}_F (\bar{M}^2, \mathbf{f}^1)$ *and* $\mathbf{f}^1 \restriction (\lambda \backslash S) = \mathbf{f}^2 \restriction (\lambda \backslash S)$,

(b) *if* $(\bar{M}^1, \mathbf{f}^2) \leq_F (\bar{M}^3, \mathbf{f}^3) \in K_\lambda^{\rm qs}$ *then* $M^2 \restriction \tau$ *cannot be* $\leq_{\mathfrak{K}}$*-embedded into* $M^3 \restriction \tau$ *over* M^1.

Proof: The proof is straightforward once you digest the meaning of weak diamond.

Let $S = \lambda$. Suppose $\langle \bar{M}^{2,\eta} : \eta \in {}^\lambda 2\rangle$ and $\mathbf{f}^2$ are as in 3.14(1), taking note of 3.15. Then we claim there is $\nu \in {}^\lambda 2$ for which 3.17 holds on taking $\bar{M}^2$ to be $\bar{M}^{2,\nu}$. Assume toward a contradiction that this fails for each ν. Then clause (b) fails, and for each $\nu \in {}^\lambda 2$ we have some $\bar{M}^{3,\nu}$ and $\mathbf{f}^{3,\nu}$ with:

$$(\bar{M}^1, \mathbf{f}^2) \leq_F (\bar{M}^{3,\nu}, \mathbf{f}^{3,\nu}) \text{ witnessed by a club } E^\nu;$$
$$f_\nu : M^{2,\nu} \restriction \tau \to M^{3,\nu} \restriction \tau \text{ over } M^1 \restriction \tau \text{ a } \leq_{\mathfrak{K}} \text{-embedding.}$$

Now by the definition of $\mathrm{WDmId}^{\mathrm{Def}}(\lambda)$ we can find $\eta_3 \neq \eta_4$ in ${}^\lambda 2$ and $\delta \in E^{\eta_3} \cap E^{\eta_4} \cap E^* \cap S$ with $E^* = \{\alpha < \lambda \text{ limit: } \beta < \alpha \text{ implies } \beta + \mathbf{f}^2(\delta), \beta + \mathbf{f}^1(\delta) < \alpha\}$, as forbidden in $(*)_2$ of 3.14(1). $\blacksquare_{3.17}$

Now we can give a reasonable non-structure theorem.

3.18 THEOREM: *Assume* $\mathbf{C}$ *is* λ*-nice,* $(\exists \mu)(\lambda = \mu^+ \ \&\ 2^\mu < 2^{\mu^+})\ \&\ 2^\lambda < 2^{\lambda^+}$, *or at least* $\mathrm{DWD}(\lambda)$, *and* $\mathrm{DWD}^+(\lambda^+)$. *Let* $J = \mathrm{WDmId}^{\mathrm{Def}}(\lambda)$.

If F *has the* (λ, J)*-coding property,* *then* $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$.

Proof: We choose by induction on $\alpha < \lambda^+$, for every $\eta \in {}^\alpha 2$, a pair $(\bar{M}^\eta, \mathbf{f}^\eta)$ such that:

(a) $(\bar{M}^\eta, \mathbf{f}^\eta) \in K_\lambda^{\rm qs}$,

(b) if $\nu \triangleleft \eta$ then $(\bar{M}^\nu, \mathbf{f}^\nu) \leq_F (\bar{M}^\eta, \bar{\mathbf{f}}^\eta)$,

(c) $(\bar{M}^\eta, \mathbf{f}^\eta) \leq^{\rm at}_{F, a_{\eta \hat{}\langle 0\rangle}} (\bar{M}^{\eta \hat{}\langle 0\rangle}, \mathbf{f}^{\eta \hat{}\langle 0\rangle})$,

(d) $(\bar{M}^\eta, \bar{\mathbf{f}}^\eta) \leq^{\rm at}_{F, a_{\eta \hat{}\langle 1\rangle}} (\bar{M}^{\eta \hat{}\langle 1\rangle}, \mathbf{f}^{\eta \hat{}\langle 1\rangle})$,

(e) if $\ell g(\eta)$ is a limit ordinal and $(\bar{M}^\eta, \bar{\mathbf{f}}^{\eta \hat{}\langle 0\rangle}) \leq_F (\bar{M}', \mathbf{f}')$ then $M^{\eta \hat{}\langle 1\rangle} \restriction \tau$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M' \restriction \tau$ over M^η,

(f) if α is limit ordinal, then $(\bar{M}^\eta, \mathbf{f}^\eta)$ is a $\leq_F$-mub of $\langle (\bar{M}^{\eta \restriction i}, \mathbf{f}^{\eta \restriction i}) : i < \alpha\rangle$.

For $\alpha = 0$ note that as $\mathbf{Seq}_\lambda \neq \emptyset$ also $\mathbf{Seq}_\lambda^s \neq \emptyset$, hence $K_\lambda^{\rm qs} \neq \emptyset$ by 3.3(d). For α limit use 3.6, 3.10. For $\alpha = \beta + 1, \beta$ a limit ordinal and $\nu \in {}^\beta 2$, define $(\bar{M}^{\nu \hat{}\langle \ell\rangle}, \mathbf{f}^{\nu \hat{}\langle \ell\rangle})$ for $\ell = 0, 1$ by 3.17. If $\alpha = \beta + 1, \beta$ non-limit, use 3.16(3). Let $M^\eta = \bigcup_{\alpha < \lambda^+} M^{\eta \restriction \alpha} \restriction \tau$ for $\eta \in {}^{(\lambda^+)}2$. Now note $\{a_{\eta \restriction (i+1)} / =^{M^\eta} : i < \lambda^+\} \subseteq M^\eta / =^{M^\eta}$ are pairwise distinct so $M^\eta \in K_{\lambda^+}$. Now we can apply 1.6 (with λ^+ here standing for λ there). $\blacksquare_{3.18}$

* * *

Unfortunately, in some interesting cases we get only weak coding.

3.19 THEOREM: *Assume* **C** *is* λ*-nice,* $(\exists\mu)(\lambda = \mu^+$ *and* $2^\mu < 2^\lambda < 2^{\lambda^+}$ *and* WDmId(λ^+) *is not* λ^+*-saturated (or at least* DfWD(λ), DfWD$^+(\lambda^+)$ *and* WDmId$^{\rm Def}(\lambda)$ *is not* λ^+*-saturated)).*

If F *has the weak* λ*-coding property (see Definition* 3.14(2)*), or at least the parallel of the conclusion of* 3.17, then $I(\lambda^+, K) \geq 2^{\lambda^+}$.

Proof: We can find $\langle S^*_\alpha : \alpha < \lambda^+\rangle$ such that:

$$S^*_\alpha \subseteq \lambda,$$
$$[\alpha < \beta \Rightarrow |S^*_\alpha \backslash S^*_\beta| < \lambda]$$

and

$$S'_\alpha =: S^*_{\alpha+1}\backslash S^*_\alpha \notin \text{ WDmId}^{\rm def}(\lambda).$$

We again choose by induction on $\alpha < \lambda^+$ for every $\eta \in {}^\alpha 2$ a pair $(\bar M^\eta, \mathbf{f}^\eta)$ such that:

(a) $(\bar M^\eta, \mathbf{f}^\eta) \in K^{\rm qs}_\lambda$,
(b) if $\nu \triangleleft \eta$ then $(\bar M^\nu, \mathbf{f}^\nu) \leq_F (\bar M^\eta, \mathbf{f}^\eta)$,
(c) $f^\eta \restriction (\lambda\backslash S_{\ell g(\eta)}) = 0_{\lambda\backslash S_{\ell g(\eta)}}$,
(d) $(\bar M^\eta, \mathbf{f}^\eta) <^{\rm at}_{F,a_{\eta\hat{}\langle 0\rangle}} (\bar M^{\eta\hat{}\langle 0\rangle}, \mathbf{f}^{\eta\hat{}\langle 0\rangle})$,
(e) $(\bar M^\eta, \mathbf{f}^\eta) <^{\rm at}_{F,a_{\eta\hat{}\langle 1\rangle}} (\bar M^{\eta\hat{}\langle 1\rangle}, \mathbf{f}^\eta)$,
(f) if $(\bar M^\eta, \mathbf{f}^{\eta\hat{}\langle 0\rangle}) \leq_F (\bar M', \mathbf{f}')$ then $M^{\eta\hat{}\langle 1\rangle} \restriction \tau$ cannot be $\leq_{\mathfrak{K}}$-embedded into $M' \restriction \tau$ over M^η,
(g) if α is a limit ordinal then $(\bar M^\eta, \mathbf{f}^\eta)$ is a $\leq_F$-mub of $\langle(\bar M^{\eta\restriction i}, \mathbf{f}^{\eta\restriction i}) : i < \alpha\rangle$.

Again there are no problems (the difference is in clause (c)). Then we apply 1.6(1) (or 1.7(1)). $\blacksquare_{3.19}$

We may like more specific sufficient conditions for many models; we explore this in 3.20, 3.21, 3.22, 3.24 which, however, are not used here.

3.20 Definition: (1) We say a λ-amalgamation choice function F for **C** has the var.[3] local λ-coding property for **C** if:

$(*)_1$ Assume $\langle M_0, M_1\rangle \in \mathbf{Seq}_2, M_0 \leq_{\mathfrak{K}^+} N_0 \in \mathfrak{K}^+_{<\lambda}, M_1 \cup N_0 \subseteq \lambda^+$, and $|M_1| \cap |N_0| = |M_0|$ and $a \in N_0, a/ =^{N_0} \notin (M_0/ =^{N_0})$ (i.e. $(\forall b \in M_0)(\neg a =^{N_0} b)$). Then we can find $N^1, N^2 \in \mathfrak{K}^+_{<\lambda}$ such that:
 (a) $N^1 = F(M_0, M_1, N_0, |N^1|, a)$,

3 There is no clear relation between "var. local" and "local" λ-coding in spite of the name.

(b) $\langle N_0, N^1\rangle \in \mathbf{Seq}_2$ and $\langle N_0, N^2\rangle \in \mathbf{Seq}_2$,

(c) $M_1 \leq_{\mathfrak{K}^+} N^1$ and $M_1 \leq_{\mathfrak{K}^+} N^2$, and $a/ =^{N^2} \in \{b/ =^{N^2} : b \in M_1\}$,

(d) $N^1 \restriction \tau, N^2 \restriction \tau$ are contradicting[4] amalgamations of $M_1 \restriction \tau, N_0 \restriction \tau$ over $M_0 \restriction \tau$; i.e. for no N', h do we have: $(N^1 \restriction \tau) \leq_{\mathfrak{K}} N' \in \mathfrak{K}_{<\lambda}$ and h is a $\leq_{\mathfrak{K}}$-embedding of N^2 into N' over $M_1 \cup N_0$; or at least

(d)$^-$ (N^1, N^2) is a τ-contradicting pair of amalgamations of M_1, N_0 over M_0 which just says: if $N^1 \leq_{\mathfrak{K}^+} N \in \mathfrak{K}^+_{<\lambda}$ then there is no $\leq_{\mathfrak{K}}$-embedding h of $N^2 \restriction \tau$ into $N \restriction \tau$ over $M_1 \cup N_0$ (i.e. is the identity on M_1 and on N_0).

(Note: This is not necessarily symmetric; and we use just the τ-reducts of N^2, M_0, M_1, N_0 so we can replace them by $N^2 \restriction \tau, M_0 \restriction \tau, M_1 \restriction \tau, N_0 \restriction \tau$ respectively.)

(2) We say that a λ-amalgamation choice function F for $\mathbf{C}$ has the *local* λ-coding property if:

$(*)_2$ if $\bar{M} = \langle M_j : j < \lambda\rangle \in \mathbf{Seq}_\lambda, \bar{N} = \langle N_j : j \leq \delta + i\rangle \in \mathbf{Seq}_{\delta+i}, a \in N_0$, $(a/ =^{N_{\delta+i}}) \notin M_{\delta+i}/ =^{M_{\delta+i}}$ and $\bar{M} \restriction (\delta+i+1) \leq^*_{\{[\delta,\delta+i]\}} \bar{N}$, and $N_{\delta+j+1} = F(M_{\delta+j}, M_{\delta+j+1}, N_{\delta+j}, |N_{\delta+j+1}|, a)$ for $j < i$, then for some $i_1, i_2 \in (i, \lambda)$ and $\bar{N}^\ell = \langle N^\ell_\alpha : \alpha < \delta + i_\ell\rangle \in \mathbf{Seq}_{\delta+i_\ell}$ for $\ell = 1, 2$ we have:

(a) $\bar{N}^\ell \restriction (\delta + i + 1) = \bar{N}$ for $\ell = 1, 2$,

(b) for $j \in [\delta + i, \delta + i_1)$ we have $N^1_{j+1} = F(M_j, M_{j+1}, N^1_j, |N^1_{j+1}|, a)$,

(c) $M_{\delta+i_2} \leq_{\mathfrak{K}^+} N^2_{\delta+i_2}$, and $a/ =^{N^2} \notin \{b/ =^{N^2} : b \in M_1\}$,

(d) $N^1_{\delta+i_1} \restriction \tau, N^2_{\delta+i_2} \restriction \tau$ are contradictory amalgamations of $M_{\delta+i_1} \restriction \tau$ and $N_\delta \restriction \tau$ over $M_\delta \restriction \tau$, or at least

(d)$^-$ $(N^1_{\delta+i_1}, N^2_{\delta+i_2})$ are τ-contradictory amalgamations of $M_{\delta+i(*)}$ and N_δ over M_δ where $i(*) = \operatorname{Min}\{i_1, i_2\}$.

(So if $i = 0$, this gives us a possibility to amalgamate, helpful for $i \in \lambda \setminus \bigcup_{\delta\in E}[\delta, \delta + \mathbf{f}(\delta)]$.)

(3) We say that a λ-amalgamation choice function F for $\mathbf{C}$ has the *weaker* local λ-coding property for $\mathbf{C}$ if:

$(*)_3$ as in part (2) but $i = 0$.

(4) In (1), (2), (3) above we say $\mathbf{C}$ has var. local or the local or the weaker local coding property respectively, if we omit the mention of F meaning for some F (clause (a) in $(*)_1$, clause (b) in $(*)_2$).

3.21 CLAIM: (1) *If F has the var. local λ-coding property for $\mathbf{C}$ or F has the local λ-coding property for $\mathbf{C}$ then F has the weaker local λ-coding property for $\mathbf{C}$.*

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4 Of course, we may consider only ones in "legal" extensions. We can also note that for the intended use, the disjointness is automatic (so 3.1(2) is not needed).

(2) *Assume*

(a) $\mathbf{C}$ *is a local* λ*-construction framework,*

(b) F *has the local* λ*-coding (or the var.* λ*-coding) (or weaker* λ*-coding) property for* $\mathbf{C}$.

Then for some F' *we have:*

(α) F'*, too, is a* λ*-amalgamation choice function for* $\mathbf{C}$,

(β) *if* $F(N_0, N_1, N_2, A, a)$ *is well defined and its* τ*-reduct is* $<_{\mathfrak{K}} M \in K_{<\lambda}$ *and* $A \subseteq A' \subseteq \lambda^+, |A'| < \lambda$, *then for some* $A'', A' \subseteq A'' \subseteq \lambda^+, |A''| < \lambda$, *and* $F'(N_0, N_1, N_2, A'', a)$ *is well defined and* $F(N_0, N_1, N_2, A, a) \leq_{\mathfrak{K}^+} F'(N_0, N_1, N_2, A'', a)$ *and* $M \leq_{\mathfrak{K}} F'(N_0, N_1, N_2, A'', a) \restriction \tau$,

(γ) F' *has the local or var. local or weaker local (respectively as in* (b)*)* λ*-coding property for* $\mathbf{C}$.

(3) *If* $\mathbf{C}$ *is local (*λ*-construction framework),* F *a* λ*-amalgamation choice function, with the var. (or just weaker) local* λ*-coding property and* $\lambda \notin WDmId(\lambda)$, *then* F *has the weak* λ*-coding property (hence under the set theoretic assumptions of* 3.19, $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$*).*

(4) *If* $\mathbf{C}$ *is local (*λ*-construction framework),* F *a* λ*-amalgamation choice function, with the local* λ*-coding property and* $\lambda \notin WDmId(\lambda)$, *then* F *has the* λ*-coding property (hence under the set theoretic assumptions of* 3.18, *we have* $I(\lambda^+, \mathfrak{K}) \geq 2^{\lambda^+}$.

Remark: The parallel of part (2) holds for local and weaker local property if F acts on sequences. See Definition 3.22 below.

Proof: (1) Check.

(2) Concerning clause (β) we use the "for technical reasons" in clause (b) of Definition 3.10(1).

(3) Like the proof of 1.4, 1.6 or of 3.23.

(4) Check. $\blacksquare_{3.21}$

In this context we may consider

3.22 Definition: We say that F is a λ-amalgamation choice function for sequences, for $\mathbf{C}$, if:

(a) if $x = F(x_1, x_2, x_3, x_4, x_5, x_6)$ is defined then for some $\alpha_1, \alpha_2, \alpha_3, \alpha < \lambda$ we have $x_\ell = \bar{M}^\ell \in \mathbf{Seq}^s_{\alpha_\ell}$ for $\ell < 3, \bar{M}^1 \lhd \bar{M}^2, t = x_4$ is a set of pairwise disjoint intervals $\subseteq \alpha_i, \bar{M}^1 <^*_t \bar{M}^3, A = x_5$ a set of $< \lambda$ ordinals $< \lambda^+$, $x = \bar{M} \in \mathbf{Seq}^s_\alpha, M_\alpha$ has universe $A, \bar{M}^1 \leq^*_{t \cup \{[\alpha_1, \alpha_2]\}} \bar{M}$,

(b) [uniqueness] as in Definition 3.10.

A companion to 3.19, with no assumption on the non-saturation of an ideal but with a weaker result is

3.23 CLAIM: *Assume:*

(a) $(\exists\mu < \lambda)(2^\mu = 2^{<\lambda} < 2^\lambda), 2^\lambda < 2^{\lambda^+}$ *or at least we have the definable weak diamond for* λ *and* λ^+*; and* $2^{<\lambda} \geq \lambda^+$.

(b) $\mathbf{C}$ *is a nice (λ-construction framework).*

(c) F *is a λ-amalgamation choice function for* $\mathbf{C}$.

(d) F *has the weaker local λ-coding property for* λ.

(e) $M^1 \in \mathbf{Seq}_\lambda$, $M^1_\lambda \restriction \tau \leq_{\mathfrak{K}} M$ *where* $M \in K_{\lambda+\ell}$.

Then letting $\mathbf{f} \in {}^{\lambda^+}\lambda^+$ *be constantly zero, for* $\bar{M}^2$ *we have* $(\bar{M}^1, \mathbf{f}) \leq_F (\bar{M}^2, \mathbf{f})$ *and* $M^2_\lambda \restriction \tau$ *cannot be* $\leq_{\mathfrak{K}}$*-embedded into* M^2.

Proof: Straightforward using 2.4.

Wlog $|M^1_\lambda| = \lambda^+$, $|M^2| = \lambda^{++}$. It is enough to choose by induction on $\alpha < \lambda^+$ a sequence $\bar{M}^\eta$ for $\eta \in {}^\alpha 2$ and ordinal $\beta_\eta < \lambda^+$ such that:

(a) $\bar{M}^\eta \in \mathbf{Seq}_\alpha$, $M^1_{\beta_\eta} \leq M^\eta_\alpha$,

(b) M^η_α has universe $(M_{\beta_\eta} \restriction [\lambda^+, \lambda^+ + \gamma_\eta))$ where $\gamma_\eta < \lambda^+$,

(c) $\nu \triangleleft \eta \Rightarrow \bar{M}^\nu \leq_F \bar{M}^\eta$ & $\beta_\nu \leq \beta_\eta$,

(d) if $\delta = \lg(\eta)$ is a limit ordinal, then $\alpha_{\beta_\eta} = \bigcup_{\gamma<\delta} \beta_{\eta\restriction\gamma}$ and $\beta_{\eta^1\langle 0\rangle} = \beta_{\eta^1\langle 1\rangle}$ and $M^{\eta^1\langle 1\rangle}_{\delta+1}$, $M^{\eta^1\langle 0\rangle}_{\delta+1}$ cannot be amalgamated over $M^1_{\beta_{\eta\wedge\langle 0\rangle}} \cup M^\eta_\delta$.

This is possible by 3.21(3).

Having the $M_\eta, \eta \in {}^{\lambda>}2$ we get the conclusion by 1.5 (see 1.2(2)). $\blacksquare_{3.23}$

3.24 Remark: (1) If we are just interested in $I(\lambda^+, K)$ rather than also in $IE(\lambda^+, K)$, then we can change the definition of τ-contradictory to:

N^1, N^2 are τ-contradictory amalgamations of M_1, N over M_0 <u>if</u> $M_0 \leq_{\mathfrak{K}^+} M_1 \leq_{\mathfrak{K}^+} N^\ell, M_0 \leq_{\mathfrak{K}^+} N_0 \leq_{\mathfrak{K}^+} N^\ell, M_0 = M_1 \cap N_0$ and there are no $N^1_*, N^2_* \in \mathfrak{K}^+_{<\lambda}$ such that: $N^\ell \leq_{\mathfrak{K}^+} N^\ell_*$ and $N^1_* \restriction \tau, N^2_* \restriction \tau$ are isomorphic over $M_1 \cup N_0$.

(2) Note that it is unreasonable to assume that we will always use the local versions: e.g. if we have a superlimit model in $\mathfrak{K}_{\lambda^+}$ and we want to have $\bar{M} \in \mathbf{Seq}_\lambda \Rightarrow M_\lambda$ superlimit, we have to add some global condition (see [Sh 600]). Also possibly we will have in the construction (i.e. in 3.18 or 3.19) that $\bar{M}_\eta$ has two "contradictory" extensions $\bar{M}_{\eta\hat{}\langle 0\rangle}, \bar{M}_{\eta\hat{}\langle 1\rangle}$ (see clauses (e) and (f) in their proof) only when $\mathrm{cf}(\ell g(\eta)) = \theta$, where $S = \{\delta : \delta < \lambda^+ \text{ and } \mathrm{cf}(\delta) = \theta\} \notin \mathrm{WDmId}(\lambda)$; or even $\ell g(\eta) \in S$, for a given $S \in \mathrm{WDmId}(\lambda)^+$. We shall deal with such cases when needed.

We intend to continue this elsewhere.

3.25 LEMMA: *Let $\mathfrak{K}$ be an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ which is categorical in λ and in λ^+, with $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}$. Assume that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, or at least that the definitional weak diamond holds for both λ^+ and λ^{++}.*

If there is a model in K_{λ^+} which is saturated over λ, <u>then</u> the minimal triples are dense in K^3_λ.

Proof: Let $\mathbf{C}$ be $\mathbf{C}^0_{\mathfrak{K},\lambda^+}$ (see Definition 3.6) hence $\mathbf{C}$ is explicitly local λ-construction framework (by 3.7(1)). Suppose toward a contradiction that above $(M^*, N^*, a^*) \in K^3_\lambda$, there is no minimal triple. We claim in this case that there is a λ^+-amalgamation choice function F for $\mathbf{C}$ with the λ^+-coding property, with domain the quadruples (M_0, M_1, M_2, A, b) such that: (M^*, N^*, a) embeds in (M_0, M_2, b); A and the universes of M_1, M_2 are contained in λ^{++}; $|A \backslash (|M_1| \cup |M_2|)| = \lambda$; and $M_0 \leq_{\mathfrak{K}} M_1, M_2$. Then applying 3.17 and 3.18 we get a contradiction.

We first make two observations concerning triples (M, N, b) lying above (M^*, N^*, b). Any such triple has the extension property by 2.11 (or just 2.9(1)) and hence we can manufacture a λ^+-amalgamation choice function with the specified domain. Furthermore, there is $M' \in K_{\lambda^+}$ with $M \leq_{\mathfrak{K}} M'$ such that $\mathrm{tp}(a, M, N)$ has more than one extension to M', by the failure of minimality.

Let us show that any λ^+-amalgamation choice function F with the specified domain has the λ^+-coding property on $\mathbf{C} = \mathbf{C}^0_{\mathfrak{K},\lambda^+}$.

Let $\bar{M}^1 \in \mathrm{Seq}_{\lambda^+}$ and let $\mathbf{f}^1 : \lambda \to \lambda$. For any set S we must find sequences $\bar{M}^{2,\eta}$ (depending on S) as in 3.14(1). The approach will be to first build suitable $M^{2,\eta}$ for all $\eta \in {}^\lambda 2$, independent of S, then restrict appropriately given S.

Let $M^1 = \bigcup_i M^1_i$. Then $M^1 \in \mathfrak{K}_{\lambda^+}$ and by our assumptions M^1 is therefore saturated over λ. Hence we may suppose $M^* \leq_{\mathfrak{K}} M^1$. We may also suppose that the universe of M^1 is an ordinal, and we may choose a subset A^2 of λ^{++} which is the union of an increasing continuous sequence A^2_α (for $\alpha < \lambda^+$) so that: $A^2_\alpha \cap |M^1| = |M^1_\alpha|$ and $A^2_0 \backslash |M^1_0|$ and $A^2_{\alpha+1} \backslash (A^2_\alpha \cup |M^1_{\alpha+1}|)$ have cardinality λ and wlog$|N^*| \setminus |M^*| = A^2_0$. Let E be the club:

$$\{\delta < \lambda : \text{ for } \alpha < \delta, \text{ we have } \alpha + \mathbf{f}^1(\alpha) + 1 < \delta\}.$$

We now define triples $(M^*_\eta, N^*_\eta, a^*)$ for all $\eta \in {}^i 2$, by induction on i, together with ordinals α_η satisfying the following conditions:

(a) for any $\eta \in {}^i 2$, the sequence $(M^*_{\eta \restriction j}, N^*_{\eta \restriction j}, a^*)$ for $j \leq i$ is increasing and continuous; and similarly the α_η are increasing and continuous.

(b) $(M^*_{<>}, N^*_{<>}, a^*) = (M^*, N^*, a^*)$.

(c) $M^*_\eta = M^1_{\alpha_\eta}$ and the universe of N^*_η is $A^2_{\alpha_\eta}$.

(d) If $\delta \in E, \eta \in {}^{\delta}2$ and $\alpha_\eta = \delta$, then:

(d1) for $i < \mathbf{f}^1(\delta)$ and $\eta \triangleleft \nu \in {}^{\delta+i+1}2$, the model N^*_ν is given by F applied to amalgamate $M^1_{\delta+i+1}$ and $N^*_{\nu\restriction(\delta+i)}$ over $M^1_{\delta+i}$, using $A^2_{\delta+i+1}$ and keeping a^* out of $M^*_{\delta+i+1}$,

(d2) for all $\nu, \nu' \in {}^{\delta+\mathbf{f}^1(\delta)+1}2$ extending η if $\nu' \neq \nu$, <u>then</u> for some β, $\mathrm{tp}(a^*, M^*_\beta, N^*_\nu) \neq \mathrm{tp}(a^*, M^*_\beta, N^*_\nu)$,

(d3) for non-zero $i \leq \mathbf{f}^1(\delta)$ and ν such that $\eta \triangleleft \nu \in {}^{\delta+i}2$ we have $M^*_\nu = M^1_{\alpha_\nu}$ and $\alpha_\nu = \alpha_{\ell g(\nu)} = \alpha_\eta + i = \alpha_\delta + i$ (so $M^*_\nu = M^1_{\delta+i}$),

(d4) for non-zero $i \leq \mathbf{f}^1(\delta)$ and $\nu, \rho \in {}^{\delta+i}2$ such that $\eta \triangleleft \nu$ & $\eta \triangleleft \rho$ we have $N^*_\nu = N^*_\rho$, call it N^1_η.

While carrying the definition the main point is guaranteeing clause (d). So let $\eta \in {}^{\delta}2$ and assume that $\alpha_\eta = \delta \in E^*$. First we define by induction on $i \leq \mathbf{f}^1(\delta)$ a model $N^1_{\eta,i}$ such that N^1_i has universe $A^2_{\delta+i}, N^1_{\eta,i}$ is $\leq_{\mathfrak{K}}$-increasing, $M^1_{\delta+i} \leq_{\mathfrak{K}} N^1_{\eta,i}$ and $N^1_{\eta,0} = N^*_\eta$ and $f, i < \mathbf{f}^1(\delta)$, then

$$N^1_{\eta,i+1} = F(M^1_{\delta+i}, M^1_{\delta+i+1}, N^1_{\eta,i}, A^2_{\delta+i+1}, a^*).$$

Next we choose by induction on $i \leq \mathbf{f}^1(\delta)$, for each $\rho \in {}^{i}2$, an ordinal $\beta_{\eta,\ell} \in [\delta + \mathbf{f}^1(\delta), \lambda^+)$ and model $N^1_{\eta,\rho}$ such that:

(i) $N^1_{\eta,\rho} \in K_\lambda$ has universe $A^2_{\beta_{\eta,\rho}}$,

(ii) $M^1_{\beta_{\eta,\rho}} \leq_{\mathfrak{K}} N^1_{\eta,\rho}$ and $(a^*/=^{N^1_{\eta,\rho}}) \notin M^1_{\beta_{\eta,\rho}}/=^{N^1_{\eta,\rho}}$,

(iii) $\beta_{\eta,<>} = \delta + \mathbf{f}^1(\delta)$,

(iv) $\rho_1 \triangleleft \rho_2 \Rightarrow \beta_{\eta,\rho_1} < \beta_{\eta,\rho_2}$ & $N^1_{\eta,\rho_1} \leq_{\mathfrak{K}} N^1_{\eta,\rho_2}$,

(v) i limit $\Rightarrow N^1_{\eta,\rho} = \bigcup_{\zeta<i} N^1_{\eta,\rho\restriction\zeta}$,

(vi) $\beta_{\eta,\rho\hat{}<0>} = \beta_{\eta\hat{}\rho\hat{}<1>}$ and

$$\mathrm{tp}(a^*, M^1_{\beta_{\eta,\rho\hat{}<0>}}, N^1_{\eta,\rho\hat{}<0>}) \neq \mathrm{tp}(a^*, M^1_{\beta_{\eta,\rho\hat{}<1>}}, N^1_{\eta,\rho\hat{}<1>}).$$

There is no problem to carry the definition. Now let

$$\begin{array}{ll} \alpha_{\eta\hat{}\rho} = \delta + i = \ell g(\eta\hat{}\rho) & \text{if } \rho \in {}^{i}2 \text{ and } i \leq \mathbf{f}^1(\delta), \\ \alpha_{\eta\hat{}\rho} = \beta_{\eta,\rho} & \text{if } \rho \in {}^{\mathbf{f}^1(\delta)+1}2, \\ M^*_{\eta\hat{}\rho} = M^1_{\alpha_{\eta\hat{}\rho}} & \text{if } \rho \in {}^{i}2,\ i \leq \mathbf{f}^1(\delta)+1, \\ N^*_{\eta\hat{}\rho} = N^1_{\eta,i} & \text{if } \rho \in {}^{i}2 \text{ and } i \leq \mathbf{f}^1(\delta), \\ N^*_{\eta\hat{}\rho} = N^1_{\eta,\rho} & \text{if } \rho \in {}^{\mathbf{f}^1(\delta)+1}1. \end{array}$$

Now check.

Having carried the induction for $\eta \in {}^{\lambda^+}2$ we let $\bar{M}^{2,\eta} = \langle N^*_{\eta\restriction\alpha} : \alpha < \lambda^+\rangle$ and $\mathbf{f}^2 = \mathbf{f}^1 + 1$. We have to check that the demand in Definition 3.14 holds.

Note that this is essentially the proof mentioned in 3.16(9).

By our initial remarks there is little difficulty in carrying out this induction. We then set $\bar{M}^{2,\eta} = \langle N^*_{\eta \restriction i} : i < \lambda^+ \rangle$ for $\eta \in {}^{\lambda^+}2$. Given a set $S \subseteq \lambda^+$ we consider $\bar{M}^{2,\eta}$ for $\eta \in {}^{\lambda^+}2$ extending $0_{\lambda \setminus S}$, together with the function $\mathbf{f}^2$ equal to $\mathbf{f}^1 + 1$ on S and to $\mathbf{f}^1$ on S. We claim that the two conditions of Definition 3.21(1) are met.

The first of these is a condition on the type of construction allowed and, of course, it has been obeyed, notably in (d1) above:

$(*)_1$ $(\bar{M}^1, \mathbf{f}^1) \leq^{\mathrm{at}}_{F,a} (\bar{M}^{2,\eta}, \mathbf{f}^1)$; and $M^{2,\eta} \restriction \alpha$ is determined by $\eta \restriction \alpha$.

The second condition referred to a club E, which can be the intersection of the club we have defined above with $\{\delta : \alpha_\delta = \delta\}$. This condition goes as follows:

$(*)_2$ it is impossible to find sequences η^3, η^4 (extending $0_{\lambda \setminus S}$), extensions $(\bar{M}^1, \mathbf{f}^2) \leq_F (\bar{M}^3, \mathbf{f}^3), (\bar{M}^4, \mathbf{f}^4)$ witnessed by clubs E^3, E^4 (i.e. E^ℓ is the intersection of the clubs which witness the atomic relations $\leq^{\mathrm{at}}_F$ implicit in $\leq_F$), and embeddings $f_\ell : M^{2,\eta^\ell} \to M^\ell$ ($\ell = 3$ or 4) over M^1 such that for some $\delta \in E \cap E^3 \cap E^4 \cap S$ we have:
  - (i) $\bar{M}^3 \restriction (\delta + \mathbf{f}^2(\delta) + 1) = \bar{M}^4 \restriction (\delta + \mathbf{f}^2(\delta) + 1)$;
  - (ii) $\mathbf{f}^3 \restriction \delta = \mathbf{f}^4 \restriction \delta$;
  - (iii) $\eta^3 \restriction \delta = \eta^4 \restriction \delta$ (call the restriction η) and $\eta^3(\delta) \neq \eta^4(\delta)$;
  - (iv) f_3, f_4 are equal on $M^{2,\eta}_\delta$; and
  - (v) for $\ell = 3, 4$, f_ℓ maps $M^{2,\eta}_\delta$ into M^ℓ_δ.

Suppose on the contrary we have η^3, η^4 (extending $0_{\lambda \setminus S}$),$(\bar{M}^1, \mathbf{f}^2) \leq_F (\bar{M}^3, \mathbf{f}^3)$, $(\bar{M}^4, \mathbf{f}^4), E^3, E^4, f_3, f_4$ and δ as above.

Let $\hat{M} = M^3_{\delta + \mathbf{f}^2(\delta)}$. It follows from condition (i) and the fact that δ belongs to the witnessing clubs E^3, E^4 that $\hat{M} = M^4_{\delta + \mathbf{f}^2(\delta)}$. Then f_3, f_4 provide embeddings of $N^*_{\eta^3 \restriction (\delta + \mathbf{f}^1(\delta) + 1)}$ and $N^*_{\eta^3 \restriction (\delta + \mathbf{f}^1(\delta) + 1)}$ into $\hat{M}$ which agrees on N^*_η (hence on a^*) and on M^1_η. By 3.16(2) we are done. $\blacksquare_{3.25}$

3.26 Lemma: *Let $\mathfrak{K}$ be an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ which is categorical in λ and in λ^+, with $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}$. Assume that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, or at least that the definitional weak diamond holds for both λ^+ and λ^{++}.*

Then:

- $(*)$ *for any $M \in K_{\lambda^+}$ and any triple (M^0, N^0, a^0) in K^3_λ with $M^0 \leq_{\mathfrak{K}} M$, we can find sequences $\bar{M} = \langle M_i : i < \lambda^+ \rangle, \bar{N} = \langle N_i : i < \lambda^+ \rangle$ such that*
  - (a) *$(M^0, N^0, a^0) = (M_0, N_0, a^0)$;*
  - (b) *(M_i, N_i, a) is increasing and continuous in K^3_λ;*
  - (c) *the union of the M_i is M;*

(d) *the set* $S(\bar{M},\bar{N},a)$ *is stationary in* λ^+*, where* $S(\bar{M},\bar{N},a)$ *is the set of* $\delta<\lambda^+$ *such that for some* $j>\delta$ *for all* $i\geq j$

if we have $(M_j,N_j,a)\leq(M_i,N^\ell,a)$ *for* $\ell=1,2$ *then we can amalgamate* N^1 *and* N^2 *over* $M\cup N_\delta$.

Proof: Otherwise, we claim that any full λ^+-amalgamation choice function will have the λ^+-coding property.

Let $\bar{M}\in \mathbf{Seq}_{\lambda^+}, \mathbf{f}:\lambda^+\to\lambda^+$ and $S\subseteq\lambda^+$ be given. Then as K is categorical in λ^+, we may suppose that M is the union of the M_i. As in the proof of 3.25 we try to define α_η, (M^*_η,N^*_η,a) with failure of amalgamation as obtained there. Now by our assumption toward contradiction for every $\eta\in{}^\lambda 2$ for some club E_η of λ^+, for every $S\in E_\eta$, defining $N^*_{\eta\restriction(\delta+\mathbf{f}(\delta)+1)}$ we have two "contradictory" amalgamations of $N^*_{\eta\restriction\delta}$ and $M_{\alpha_\eta\restriction(\delta+\mathbf{f}(\delta)+1)}$. So as there we get $I(\lambda^{++},K)=2^{\lambda^{++}}$ contradicting an assumption. $\blacksquare_{3.26}$

3.27 Lemma: *Let* $\mathfrak{K}$ *be an abstract elementary class with* $LS(\mathfrak{K})\leq\lambda$ *which is categorical in* λ,λ^+ *and* λ^{++} *and with no model in cardinality* λ^{+3}.

Suppose that there is no model in K_{λ^+} *saturated above* λ *and that:*

($*'$) *for any* $M\in K_{\lambda^+}$ *and any triple* (M^0,N^0,a^0) *in* K^3_λ *with* $M^0\leq_{\mathfrak{K}}M$*, we can find sequences* $\bar{M}=\langle M_i:i<\lambda^+\rangle,\bar{N}=\langle N_i:i<\lambda^+\rangle$ *such that*

(a) $(M^0,N^0,a^0)=(M_0,N_0,a^0)$;

(b) (M_i,N_i,a) *is increasing and continuous in* K^3_λ;

(c) *the union of the* M_i *is* M;

(d) *the set* $S(\bar{M},\bar{N},a)$ *is stationary in* λ^+*, where* $S(\bar{M},\bar{N},a)$ *is the set of* $\delta<\lambda^+$ *such that for some* $j>\delta$ *for all* $i\geq j$

if we have $(M_j,N_j,a)\leq(M_i,N^\ell,a)$ *for* $\ell=1,2$ *then we can amalgamate* N^1 *and* N^2 *over* $M_i\cup N_\delta$.

__Then__ the minimal triples are dense in K^3_λ.

Proof: Suppose that there is no minimal triple above (M^*,N^*,a^*). It suffices to show that there is no maximal model in $K_{\lambda^{++}}$ and, as $K_{\lambda^{++}}$ is categorical, this will follow from the existence of a single pair of models (M',N') in $K_{\lambda^{++}}$ with $M'<_{\mathfrak{K}}N'$. So it suffices to show:

every triple (M,N,a) in $K^3_{\lambda^+}$ has a proper extension in $K^3_{\lambda^+}$,

as the desired pair (M',N') can then be built as the limit of an increasing continuous chain.

Fix (M,N,a) in $K^3_{\lambda^+}$. As there is no model saturated over λ in K_{λ^+}, there is some M_0 in K_λ over which there are more than λ^+ types. By λ-categoricity we may suppose $M_0 \leq_{\mathfrak{K}} M$. Fix a triple (M_0, N_0, b) in K^3_λ for which $\mathrm{tp}(b, M_0, N_0)$ is not realized in M.

Apply $(*')$ to M and (M_0, N_0, b) to get sequences $\overline{M}^0, \overline{N}^0$ of length λ^+ as in $(*')$ and $(M_0, N_0) = (M^0_0, N^0_0)$. Let $S = S(\overline{M}^0, \overline{N}^0, b)$. As $\mathfrak{K}$ is categorical in λ, wlog $M = M^0 = \bigcup M^0_i$. Let $M^1 = \bigcup N^0_i$. As $\mathfrak{K}$ has amalgamation in λ^+ we may suppose M^1, $N \leq_{\mathfrak{K}} N^1$ with $N^1 \in \mathfrak{K}_{\lambda^+}$.

We can also choose an increasing continuous sequence (M'_i, N'_i, a^*) for $i < \lambda^+$ beginning with (M^*, N^*, a^*) such that each (M'_i, N'_i, a^*) is reduced and $\mathrm{tp}(a^*, M'_i, N'_i)$ has more than one extension in $\mathcal{S}(M'_{i+1})$, using the failure of minimality and 2.7(1). By categoricity we may suppose $M = \bigcup M'_i$. Set $M^2 = \bigcup N'_i$. By amalgamation we may suppose $M^2, N^1 \leq_{\mathfrak{K}} N^2 \in K_{\lambda^+}$.

We claim that one of the triples (M^1, N^1, a) or (M^2, N^2, a) is a proper extension of (M, N, a) as $(M, N, a) \leq (M^\ell, N^\ell)$ this means that $M \neq M^1$ or $M \neq M^2$. Suppose on the contrary that a belongs to both M^1 and M^2.

Represent N^2 as the union of a continuous $\leq_{\mathfrak{K}}$-increasing chain $\langle N^*_i : i < \lambda^+\rangle$ of models in K_λ. Let E be

$$\{i < \lambda^+ : M^0_i = M'_i; N^*_i \cap M = M^0_i; N^*_i \cap M^1 = N^0_i; N^*_i \cap M^2 = N'_i\};$$

it is a club in λ^+.

Fix $\delta \in E \cap S$ such that a is in N^0_δ and N'_δ; exists as S is stationary and $a \in M^1 \cap M^2$. We show now that $a^* \in N^0_\delta$. Now $(M'_\delta, N'_\delta, a^*)$ is reduced. If $a^* \notin N^0_\delta$ then $(N^0_\delta, N^*_\delta, a^*)$ lies above $(M'_\delta, N'_\delta, a^*)$ and hence by the latter being reduced $N^0_\delta \cap N'_\delta \subseteq M'_\delta$; but the element a witnesses the failure of this condition noting $a \notin M'_\delta$ as $a \notin M$. So $a^* \in N^0_\delta$.

Let $j > \delta$ be chosen in accordance with the definition of $S(\bar{M}, \bar{N}, b)$ and let $i > j' > j$ be such that $j' \in E$. As $\mathrm{tp}(a^*, M'_{j'}, N'_{j'})$ has more than one extension to $M'_{j'+1}$, the same applies to M'_i. However, $M'_i = M^0_i$ and $M'_{j'} = M^0_{j'}$ and thus $\mathrm{tp}(a^*, M^0_{j'}, N^2) = \mathrm{tp}(a^*, M^0_{j'}, N^*_{j'}) = \mathrm{tp}(a^*, M^0_{j'}, N^0_{j'})$ has more than one extension over M^0_i. Thus, M^0_i and $N^0_{j'}$ may be amalgamated in two incompatible ways over $M^0_{j'}$, getting N^+ and N^-, say (in fact, moreover, the models N^+ and N^- cannot be amalgamated over M_i preserving the images of a^*). Furthermore both (M^0_i, N^+, b) and (M^0_i, N^-, b) lie above (M^0_j, N^0_j, b) in $K^3_{\lambda^+}$, that is, b is not mapped into M^0_i, because M does not realize $\mathrm{tp}(b, M_0, N_0)$. However, this contradicts the definition of S, as the triples (M^0_i, N^+, b) and (M^0_i, N^-, b) cannot be amalgamated over (M^0_i, N^0_δ) since a^* belongs to N^0_δ. $\blacksquare_{3.27}$

3.28 THEOREM: *Let $\mathfrak{K}$ be an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ which is categorical in λ and in λ^+ with $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}$. Assume that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, or at least that the definitional weak diamond holds for both λ^+ and λ^{++}.*

Then under either of the following assumptions, the minimal triples are dense in K^3_λ:

(A) *K is categorical in λ^{++} and has no model in cardinality λ^{+3};*

(B) *there is a model saturated above λ in cardinality λ^+.*

Proof: If assumption (B) holds, use 3.25; so assume (A). Now by the previous lemmas 3.26, 3.27 the conclusion follows. Note that $(*')$ of 3.27 is exactly the negation of $(*)$ of 3.26. $\blacksquare_{3.28}$

3.29 Remark: This will be proved without the additional assumptions (A, B) in [Sh 603]. In any case this does not affect the proof of Theorems 0.2, 0.3.

3.30 CLAIM: *Let $\mathfrak{K}$ be an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ which is categorical in λ and in λ^+, with $1 \leq I(\lambda^{++}, K) < 2^{\lambda^{++}}$, and with no model in cardinality λ^{+3}. Assume that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.*

<u>Then</u> the minimal triples are dense in K^3_λ.

Proof: If $2^{\lambda^+} > \lambda^{++}$ we get the conclusion by 2.7. If $2^{\lambda^+} = \lambda^{++}$, then as $2^\lambda < 2^{\lambda^+}$ we have $2^\lambda = \lambda^+$. Thus, there is a model in K_{λ^+} which is saturated above λ, and Lemma 3.28 applies. $\blacksquare_{3.30}$

4. Minimal types

We return to the analysis of minimal types initiated in 2.12. We use from §2 only 2.1, 2.6 and 2.9, so there are repetitions.

4.1 HYPOTHESIS:

(a) $\mathfrak{K}$ is an abstract elementary class with $LS(\mathfrak{K}) \leq \lambda$ (for simplicity $K_{<\lambda} = \emptyset$).

(b) $\mathfrak{K}$ is categorical in λ, λ^+ with $K_{\lambda^{+2}} \neq \emptyset$ (note: (a) + (b) = $(*)^3_\lambda$ of 2.4).

(c) $\mathfrak{K}$ has amalgamation in λ (2.2(1)), so by (a)+(c), we have $(*)^2_\lambda$ from 2.9, hence $\mathfrak{K}$ satisfies the model theoretic properties which were deduced in 2.4–2.6 and 2.9 in particular:
  - (i) every $(M, N, a) \in K^3_\lambda$ has the weak extension property 2.4;
  - (ii) criteria for the extension property 2.9;
  - (iii) basic definitions and properties 2.3, 2.6.

4.2 Definition: (1) If $p \in \mathcal{S}(N), N \in K_\lambda$ and $N' \in K_\lambda$ remember (from Definition 2.13)

$$\mathcal{S}_p(N') = \{f(p) : f \text{ is an isomorphism from } N \text{ onto } N'\}$$

and let

$$\mathcal{S}_{\geq p}(N') = \{q \in \mathcal{S}(N') : q \text{ not algebraic (i.e. not realized by any } c \in N') \text{ and, for some } N'' \in K_\lambda,\ N'' \leq_{\mathfrak{K}} N', \text{ we have } q \restriction N'' \in \mathcal{S}_p(N'')\}.$$

(2) We say the type $p \in \mathcal{S}(N)$ is **λ-algebraic** if $\|N\| \leq \lambda$, and for every M such that $N \leq_{\mathfrak{K}} M$ we have $\lambda \geq |\{c \in M : \text{tp}(c, N, M) = p\}|$.

4.3 CLAIM: *If $(M_0, M_1, a) \in K^3_\lambda$ is minimal, <u>then</u> it has the extension property.*

Proof: Let $p^* = \text{tp}(a, M_0, M_1)$, and assume it is a counter-example. We note:

$\bigotimes_1$ For some M^* we have $M_0 \leq_{\mathfrak{K}} M^* \in K_\lambda$ but for no M^+ and b do we have $M^* \leq_{\mathfrak{K}} M^+ \in K_\lambda, b \in M^+ \setminus M^*$ and b realizes p^*.
[Why? If not for every $N, M_0 \leq_{\mathfrak{K}} N \in K_\lambda$, we can find N_1, $N \leq_{\mathfrak{K}} N_1 \in K_\lambda$ and $b \in N_1 \setminus N$ which realizes p^*. Hence (as $\mathfrak{K}_\lambda$ has amalgamation in λ) we can find N_2 such that $N_1 \leq_{\mathfrak{K}} N_2 \in K_\lambda$, and g a $\leq_{\mathfrak{K}}$-embedding of M_1 into N_2 extending id_{M_0} such that $g(a) = b$. This proves the extension property.]

$\bigotimes_2$ If $p \in \mathcal{S}_{\geq p^*}(N)$ and $N \in K_\lambda$ and $N \leq_{\mathfrak{K}} N^* \in K$, <u>then</u> the set of elements of $b \in N^*$ realizing p has cardinality $\leq \lambda$.
[Why? by 4.1(c)(ii); so indirectly 2.9(2).]

$\bigotimes_3$ If $N \in K_\lambda$, then $|\mathcal{S}_{\geq p^*}(N)| > \lambda^+$.

Proof of $\bigotimes_3$: If N forms a counterexample, as K is categorical in λ and using $\bigotimes_2$ we can find $\langle N_i : i < \lambda^+ \rangle$, $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of K_λ such that:

(*) for every $\alpha < \lambda^+$ and $q \in \mathcal{S}_{\geq p^*}(N_\alpha)$, for some $\beta = \beta_q < \lambda^+$ we have: for no N', b do we have $N_\beta \leq_{\mathfrak{K}} N' \in K_\lambda, b \in N' \setminus N_\beta$ and b realizes q.

So $N_{\lambda^+} = \bigcup_{i<\lambda^+} N_i$ has the property

(**) if $\bar{N}' = \langle N'_\alpha : \alpha < \lambda^+ \rangle$ is a representation of N_{λ^+}, then for a club of $\delta < \lambda^+$ for every $q \in \mathcal{S}_{\geq p^*}(N'_\delta)$ for a club of $\beta \in (\delta, \lambda^+)$, for no N', b do we have: $N_\beta \leq_{\mathfrak{K}} N' \in K_\lambda, b \in N' \setminus N_\beta$ and b realizes q.

On the other hand, we can choose by induction on $\alpha < \lambda^+$ a triple $(N_{0,\alpha}, N_{1,\alpha}, a) \in K^3_\lambda$ increasing continuous in α such that $(N_{0,0}, N_{1,0}, \alpha) = (M_0, M_1, a)$ and $N_{0,\alpha} \neq N_{0,\alpha+1}$ (existence by the weak extension property; i.e. 2.4 = 4.1(c)(i)).

Now $N_{0,\lambda^+} = \bigcup_{\alpha<\lambda^+} N_{0,\alpha} \in K_{\lambda^+}$ does not satisfy the statement (**): $\langle N_{0,\alpha} : \alpha < \lambda^+ \rangle$ is a representation of N_{0,λ^+}, and for every α, $\text{tp}(a, N_{0,\alpha}, N_{1,\alpha})$ extend $\text{tp}(a, N_{0,0}, N_{1,0}) = \text{tp}(a, M_0, M_1) = p^*$, hence $\text{tp}(a, N_{0,\alpha}, N_{1,\alpha}) \in \mathcal{S}_{\geq p^*}(N_{0,\alpha})$ satisfies: for every $\beta \in (\alpha, \lambda^+)$, there is $N', N_{0,\beta} \leq_{\mathfrak{K}} N'$ and some $b \in N' \setminus N_{0,\beta}$ realizes $\text{tp}(a, N_{0,\alpha}, N_{1,\alpha})$; simply choose $(N', b) = (N_{1,\beta}, a)$.

So $N_{0,\lambda^+}, N_{\lambda^+}$ cannot be isomorphic (as one satisfies (**) the other not). But both are in K_{λ^+}, contradicting the categoricity of $\mathfrak{K}$ in λ^+. $\blacksquare_{\otimes_3}$

To finish the proof of 4.3 it is enough to prove

4.4 CLAIM: *If $p^* \in \mathcal{S}(M_0)$ is minimal, $M_0 \in K_\lambda$, <u>then</u> $N \in K_\lambda \Rightarrow |\mathcal{S}_{\geq p^*}(N)| \leq \lambda^+$.*

Proof: By 4.6 and 4.7 below. Note that $\mathcal{S}_{\geq p^*}(N)$ has the same cardinality for every $N \in K_\lambda$.

4.5 CLAIM: (1) *If $N_1 \leq_{\mathfrak{K}} N_2$ are in K_λ and $p_1 \in \mathcal{S}(N_1)$ is minimal and is omitted by N_2 <u>then</u> p_1 has a unique extension in $\mathcal{S}(N_2)$, call it p_2, and $p_1 \in \mathcal{S}_{\geq p^*}(N_1) \Rightarrow [p_2 \in \mathcal{S}_{\geq p^*}(N_2)$ and p_2 is minimal].*
(2) *If $N_1 \leq_{\mathfrak{K}} N_2$ are in $K_\lambda, p_1 \in \mathcal{S}(N_1)$ minimal, <u>then</u> p_1 has at most one non-algebraic extension in $\mathcal{S}(N_2)$ called p_2, it is minimal and $p_1 \in \mathcal{S}_{\geq p^*}(N_1) \Rightarrow p_2 \in \mathcal{S}_{\geq p^*}(N_2)$.*
(3) *(Continuity) If $\langle N_i : i \leq \alpha\rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of $K_\lambda, p_i \in \mathcal{S}(N_i), p_0$ minimal, $p_i \in \mathcal{S}(N_i)$ extends p_0 and is non-algebraic, then $\langle p_i : i \leq \alpha\rangle$ is increasing continuously.*

Proof of 4.5: Easy. E.g.,
(3) If $i < j \leq \alpha$ then $p_j \restriction N_i$ is well defined, it belongs to $\mathcal{S}(N_i)$, also it is non-algebraic and extending p_0, hence by the uniqueness (=4.5(1)) we have $p_i = p_j \restriction N_i$. If $\delta \leq \alpha, p_\delta \in \mathcal{S}(N_\delta)$ extends p_i for $i < \delta$; if $p'_\delta \in \mathcal{S}(N_\delta)$ extends each p_i $(i < \delta)$ then it extends p_0 and is non-algebraic, hence by uniqueness $p'_\delta = p_\delta$. $\blacksquare_{4.5}$

4.6 CLAIM: *If $N \in K_\lambda, S \subseteq \mathcal{S}(N)$ and $|S| > \lambda^+$, <u>then</u> we can find N^*, N_i in K_λ (for $i < \lambda^{++}$) such that:*
- (α) *$N \leq_{\mathfrak{K}} N^* <_{\mathfrak{K}} N_i$,*
- (β) *for no $i_0 < i_1 < \lambda^{++}$ and $c_\ell \in N_{i_\ell} \backslash N^*$ (for $\ell = 0, 1$) do we have $tp(c_0, N^*, N_{i_0}) = tp(c_1, N^*, N_{i_1})$*
- (γ) *there are $a_i \in N_i$ (for $i < \lambda^{++}$) such that $tp(a_i, N, N_i) \in S$ is not realized in N^* (and they are pairwise distinct).*

Remark: We use here less than Hypothesis 4.1:
- (*) $\mathfrak{K}$ is abstract elementary class with amalgamation in λ, categorical in λ, $K_{\lambda^+} \neq \emptyset$.

The same applies to 4.7.

Proof: Without loss of generality $|N| = \lambda$; now choose by induction on $\alpha < \lambda^{++}, \bar{N}^\alpha, N_\alpha, a_\alpha$ such that:

(A) $N_\alpha \in K_{\lambda^+}$ has a set of elements $\lambda \times (1+\alpha)$ and N_α is $\leq_{\mathfrak{K}}$-increasing continuous in α,

(B) $\bar{N}^\alpha = \langle N_i^\alpha : i < \lambda^+ \rangle$ is a representation of N_α (i.e. is $\leq_{\mathfrak{K}}$-increasing continuous, $\|N_i^\alpha\| \leq \lambda$ and $N_\alpha = \bigcup_{i<\lambda^+} N_i^\alpha$),

(C) for $\alpha < \lambda^{++}$ successor, if $i < j < \lambda^+, p \in \mathcal{S}(N_i^\alpha)$ is realized in N_j^α and is λ-algebraic (see Definition 4.2(2)) then for no N', b do we have $N_j^\alpha \leq_{\mathfrak{K}} N' \in K_\lambda$ and $b \in N' \backslash N_j^\alpha$ realizes p (actually not needed),

(D) $N = N_0^0 \leq_{\mathfrak{K}} N_0$ and $a_\alpha \in N_{\alpha+1} \backslash N_\alpha$ realizes some $p_\alpha \in \mathcal{S}$ not realized in N_α,

(E) if $\aleph_0 < \operatorname{cf}(\alpha) \leq \lambda$ then for $j < \lambda^+$ let $M_j^\alpha = \bigcup_{\beta \in C} N_j^\beta$; if C is club of α such that any $\beta_1 < \beta_2$ from C, $N_j^{\beta_1} = N_j^{\beta_2} \cap N_{\beta_1}$ (any two such C's give the same result),

(F) for each $\alpha < \lambda^{++}$, for a club E_α^0 of ordinals $i < \lambda^+$ we have $(N_i^\alpha, N_i^{\alpha+1}, a_\alpha)$ is "almost reduced", that is:

$(*)_i$ for every ($i \in E_\alpha^0$ and) $b \in N_i^{\alpha+1} \backslash N_i^\alpha$ the type $\operatorname{tp}(b, N_i^\alpha, N_i^{\alpha+1})$ is not realized in N_α (a key point).

There is no problem to carry out the construction (possible $\mathfrak{K}$ has amalgamation in λ and, concerning clause (F), as $tp(a_\alpha, N_i^\alpha, N_i^{\alpha+1})$ extends $tp(a_\alpha, N, N_i^{\alpha+1})$ which is not realized in N_α). Let $w_i^\alpha =: \{\beta : N_i^{\alpha+1} \cap N_{\beta+1} \not\subseteq N_\beta\}$, so necessarily $|w_i^\alpha| \leq \lambda, w_i^\alpha$ is increasing continuous in $i < \lambda^+$ and $\alpha = \bigcup_{i<\lambda^+} w_i^\alpha$ and for $\beta < \alpha$ let

$$\mathbf{i}(\beta, \alpha) = \operatorname{Min}\{i : \beta \in w_i^\alpha\}.$$

Now for every $\alpha \in S^* =: \{\delta < \lambda^{++} : \operatorname{cf}(\delta) = \lambda^+\}$, the set

$$\begin{aligned} E_\alpha =: \{i < \lambda^+ : i \text{ limit }, N \leq_{\mathfrak{K}} N_i^{\alpha+1}, a_\alpha \in N_i^{\alpha+1}, &\text{ and for every } \beta < \alpha \\ \text{if } \beta \in w_i^\alpha \text{ then } N_i^\beta = N_i^\alpha \cap N_\beta &\text{ and for } j < i \\ \text{the closure of } w_j^\alpha \text{ (in } \alpha) &\text{ is included in } w_i^\alpha \text{ and} \\ \beta_1 < \beta_2 \;\&\; \beta_1 \in w_j^\alpha \;\&\; \beta_2 \in w_j^\alpha &\Rightarrow \mathbf{i}(\beta_1, \beta_2) < i\} \end{aligned}$$

is a club of λ^+.

As we can assume $\lambda > \aleph_0$ (ignoring $\lambda = \aleph_0$ as was treated earlier in [Sh 88] though for a $\mathrm{PC}_{\aleph_0}$ class, or see [Sh 603], §4), we can choose $j_\alpha \in E_\alpha$ such that $\operatorname{cf}(j_\alpha) = \aleph_1$ and let $\delta_\alpha = \sup(w_{j_\alpha}^\alpha)$; now $w_{j_\alpha}^\alpha$ is closed under ω-limits (as $\langle w_j^\alpha : j \leq \alpha \rangle$ is increasing continuous, $j < \alpha \Rightarrow \operatorname{closure}(w_j^\alpha) \subseteq w_{j+1}^\alpha$) and $\aleph_1 = \operatorname{cf}(\operatorname{otp} w_{j_\alpha}^\alpha)$, so there is $\langle \beta_\varepsilon : \varepsilon < \omega_1 \rangle$ increasing continuous with limit

$\delta_\alpha, \beta_\varepsilon \in w^\alpha_{j_\alpha}$, so $\varepsilon < \zeta < \omega_1 \Rightarrow N^{\beta_\varepsilon}_j = N^{\beta_\zeta}_j \cap N_{\beta_\varepsilon}$ and easily $N^\alpha_j \cap N_{\beta_\varepsilon} = N^{\beta_\varepsilon}_j$, hence

$$\bigoplus \qquad M^{\delta_\alpha}_{j_\alpha} = \bigcap \Big\{ \bigcup_{j \in C} N^\beta_j : C \text{ a club of } \delta_\alpha \Big\}$$

so $N^\alpha_{j_\alpha} = M^{\delta_\alpha}_{j_\alpha}$ (see [Sh 351, §4]).

By the Fodor lemma for some j^*, α^* and stationary $S \subseteq S^*, \alpha \in S^* \Rightarrow j_\alpha = j^*$ & $\delta_\alpha = \delta^*$. So for all $\alpha \in S, N^\alpha_{j_\alpha}$ are the same, say N^*. So $N^* \in K_\lambda$, for $\alpha \in S$, $q_\alpha =$ tp$(a_\alpha, N^*, N^{\alpha+1}_{j_\alpha})$ extend $p_\alpha (\in \mathcal{S})$. Also, if $r \in \mathcal{S}(N^*)$ is realized in $N^{\alpha+1}_{j_\alpha}$, say by b (for some $\alpha \in S$), then no member of $\bigcup\{N^{\beta+1}_{j^*} \backslash N^\beta_{j^*} : \beta \in S \cap \alpha\}$ realizes it (holds by clause (G)). So the sets $\Gamma_\alpha = \{\text{tp}(b, N^{\delta^*}_{j^*}, N^{\alpha+1}_{j^*}) : b \in N^{\alpha+1}_{j^*} \backslash N^*\}$ for $\alpha \in S$ are pairwise disjoint and each has a member extending $p_\alpha \in \mathcal{S}$ (as exemplified by a_α and p_α is not extended by any $p \in \bigcup_{\beta<\alpha} \Gamma_\beta$ (as p_α is not realized in N_α)). $\blacksquare_{4.6}$

4.7 Claim: *Assume p^* is a counterexample to 4.4.*
(1) *If $N \in K_\lambda, \Gamma \subseteq \mathcal{S}_{\geq p^*}(N), |\Gamma| \leq \lambda^+$ then*

$$\{p \in \mathcal{S}_{\geq p^*}(N) : \textit{for some } N', N \leq_{\mathfrak{K}} N' \in K_\lambda \textit{ and some } b \in N' \textit{ realizes } p \textit{ but no } b \in N' \textit{ realizes any } q \in \Gamma\}$$

has cardinality $\geq \lambda^{++}$.
(2) *We can find $N \in K_{\lambda^+}$, and $N_i, N <_{\mathfrak{K}} N_i \in K_{\lambda^+}$ for $i < 2^{\lambda^+}$ such that the set $\Gamma_i = \{tp(a, N, N_i) : a \in N_i\}$ are pairwise distinct, in fact, no one embeddable into another (so we get $I(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$ and if $(2^\lambda)^+ < 2^{\lambda^+}$ then $IE(\lambda^+, \mathfrak{K}) = 2^{\lambda^+}$, thus contradicting the categoricity in λ^+ from the assumptions).*

Proof: (1) Apply 4.6 with N and $\mathcal{S}_{\geq p^*}(N)$ here standing for N and $\mathcal{S}$ there, so we get N^*, N_i $(i < \lambda^{++})$ and M^* such that $N^* \leq_{\mathfrak{K}} N_i \in \mathfrak{K}_\lambda$, and $\Gamma_i = \{\text{tp}(a, N^*, N_i) : a \in N_i \backslash N^*\}$ are pairwise disjoint and there are $p_i \in \Gamma_i, p_i \restriction N \in \mathcal{S}_{\geq p^*}(N)$ pairwise distinct; now p_i is not algebraic, hence $p_i \in \mathcal{S}_{\geq p^*}(N^*)$. As K is categorical in λ, without loss of generality $N^* = N$, so all but $\leq \lambda^+$ of the models N_i can serve as the required N'.
(2) Now by part (1) of 4.7 we can choose, by induction on $i < \lambda^+$, $\langle (N_\eta, \Gamma_\eta) : \eta \in {}^i 2 \rangle$ such that:

(a) $N_\eta \in K_\lambda$ and $\Gamma_\eta \subseteq \bigcup_{j<i} \mathcal{S}_{\geq p^*}(N_{\eta \restriction j})$ and $|\Gamma_\eta| \leq \lambda$,
(b) if $\nu \triangleleft \eta$ then $N_\nu \leq_{\mathfrak{K}} N_\eta$ and $\Gamma_\eta \subseteq \Gamma_\nu$,
(c) some $p \in \Gamma_{\eta\hat{}\langle 0 \rangle}$ is from $\mathcal{S}(N_\eta)$ and is realized in $N_{\eta\hat{}\langle 1 \rangle}$ (similarly for $\Gamma_{\eta\hat{}\langle 1 \rangle}, N_{\eta\hat{}\langle 0 \rangle}$),

(d) if i is a limit ordinal, then $N_\eta = \bigcup_{j<i} N_{\eta\restriction j}$ and $\Gamma_\eta = \bigcup_{j<i} \Gamma_{\eta\restriction j}$.

The successor case is done by 4.7(1) (you may object that the type in Γ_η is not from $\mathcal{S}_{\geq p^*}(N_\eta)$ but from $\bigcup_{j<i} \mathcal{S}_{\geq p^*}(N_{\eta\restriction j})$; however, they are minimal—see 4.5(1)). For $\eta \in {}^{\lambda^+}2$ let $N_\eta = \bigcup_{i<\lambda^+} N_{\eta\restriction i}$. Now by 1.4(2), $\{N_\eta/\cong : \eta \in {}^{\lambda^+}2\}$ has cardinality 2^{λ^+}, contradiction. Concerning IE, see 1.6(1). $\blacksquare_{4.7}$, $\blacksquare_{4.4}$, $\blacksquare_{4.3}$

4.8 CLAIM: (1) *If $M_0 \in K_\lambda, M_1 \in K_{\lambda^+}$, and $M_0 \leq_{\mathfrak{K}} M_1$ <u>then</u> every minimal $p \in \mathcal{S}(M_0)$ is realized in M.*

(2) *Every $M_1 \in K_{\lambda^+}$ is saturated at least for minimal types (i.e. if $M_0 \leq_{\mathfrak{K}} M_1$, $M_0 \in K_\lambda$ and $M_1 \in K_{\lambda^+}$ <u>then</u> every minimal $p \in \mathcal{S}(M_0)$ is realized in M_1).*

(3) *If $M \in K_\lambda$ then $\{p \in \mathcal{S}(M) : p$ minimal$\}$ has cardinality $\leq \lambda^+$.*

Proof: (1) Let $\bar{N} = \langle N_\alpha : \alpha < \lambda^+\rangle$ be a representation of $N_{\lambda^+} \in K_{\lambda^+}$. Let $N \in K_\lambda, p \in \mathcal{S}(N)$ be minimal. We ask:

$(*)_p$ is there a club of $\alpha < \lambda^+$ such that every $q \in \mathcal{S}_{\geq p}(N_\alpha)$ is realized in λ^+?

By 4.4 there is $N'_{\lambda^+} \in K_{\lambda^+}$ for which the answer is yes, hence, as $\mathfrak{K}$ is categorical in λ^+, this holds for N_{λ^+}. So this holds for every minimal p. Now if $N' \leq_{\mathfrak{K}} N_{\lambda^+}, N' \in K_\lambda$ and $p \in \mathcal{S}(N')$ is minimal then for some $\alpha, N' \leq_{\mathfrak{K}} N_\alpha$ and for every $\beta \in [\alpha, \lambda^+)$, p has a unique non-algebraic extension $p_\beta \in \mathcal{S}(N_\beta)$ (which necessarily is minimal, exists by 2.1). Now $p_\beta \in \mathcal{S}_{\geq p}(N_\beta)$, hence for a club of $\beta < \lambda^+, p_\beta$ is realized in N_{λ^+}, so we have finished the proof of part (1).

(2) By part (1).

(3) Follows by part (1). $\blacksquare_{4.8}$

From 4.3, 2.9(1) we can conclude

4.9 CONCLUSION: Every $(M_0, M_1, a) \in K^3_\lambda$ has the extension property.

5. Inevitable types and stability in λ

5.1 HYPOTHESIS: Assume the model theoretic assumptions from 4.1 and

(d) there is a minimal member of K^3_λ (follows from the conclusion of 3.30).

5.2 Definition: We call $p \in \mathcal{S}(N)$ **inevitable** if: $N \leq_{\mathfrak{K}} M$ & $N \neq M \Rightarrow$ some $c \in M$ realizes p. We call $(M, N, a) \in K^3_\lambda$ **inevitable** if $\text{tp}(a, M, N)$ is inevitable.

Now using 4.3–4.8 we shall deduce

5.3 CLAIM: (1) *If there is a minimal triple in* K^3_λ, *then there is an inevitable* $p = tp(a, N, N_1)$ *with* $(N, N_1, a) \in K^3_\lambda$ *minimal.*
(2) *Moreover, if* $p_0 \in \mathcal{S}(N_0)$ *is minimal,* $N_0 \in K_\lambda$ *then we can find* $N_1, N_0 \leq_{\mathfrak{K}} N_1 \in K_\lambda$ *such that the unique non-algebraic extension* p_1 *of* p_0 *in* $\mathcal{S}(N_1)$ *is inevitable.*

Proof of 5.3: (1) Follows by part (2).
(2) Let $(M_0, M_1, a) \in K^3_\lambda$ be minimal and $p_0 = \text{tp}(a, M_0, M_1)$. We try to choose by induction on i a model N_i such that: $N_0 = M_0, N_i \in K_\lambda$ is $\leq_{\mathfrak{K}}$-increasing continuously and N_i omits $p_0, N_i \neq N_{i+1}$. If we succeed, $\bigcup_{i<\lambda^+} N_i$ is a member of K_{λ^+} which is non-saturated for minimal types, contradicting 4.8(2). As for $i = 0$, i limit we can define, necessarily for some i we have N_i but not N_{i+1}. Now p_0 has a unique non-algebraic extension in $\mathcal{S}(N_i)$ which we call p_i and p_0 has no algebraic extension in $\mathcal{S}(N_i)$. [Why? As N_i omits p_0.] So p_i is the unique extension of p_0 in $\mathcal{S}(N_i)$ [by 4.5(1)], and so

(*) if $N_i \leq_{\mathfrak{K}} N' \in K_\lambda$ and $N' \neq N$, then p_i is realized in N'.

By L.S. we can omit "$N' \in K_\lambda$", so (N_i, p_i) are as required. $\blacksquare_{5.3}$

5.4 FACT: Inevitable types have few ($\leq \lambda$) conjugates (i.e. for $p \in \mathcal{S}(M_0)$ inevitable $M_0 \in K_\lambda, M_1 \in K_\lambda$ we have $|\mathcal{S}_p(M_1)| \leq \lambda$), moreover $|\{p \in \mathcal{S}(N) : p \text{ inevitable}\}| \leq \lambda$ for $N \in K_\lambda$.

Proof: Easy.

The following construction shall play a central role in this paper.

5.5 CLAIM: *For any limit* $\alpha < \lambda^+$, *we can find* $\langle N_i : i \leq \alpha \rangle$ *and* $\langle p_i : i \leq \alpha \rangle$ *such that:*

(i) $N_i \in K_\lambda$,
(ii) N_i *is* $\leq_{\mathfrak{K}}$*-increasing continuous,*
(iii) $p_i \in \mathcal{S}(N_i)$ *is minimal,*
(iv) p_i *increases continuously (see* 4.5(3)*),*
(v) p_0 *is inevitable,*
(vi) p_α *is inevitable,*
(vii) $N_i \neq N_{i+1}$, *moreover some* $c \in N_{i+1} \setminus N_i$ *realizes* p_0 *(hence* p_i*).*

Remark: Why not just try to build a non-saturated model in order to prove 5.5? It works, too.

Proof: Choose $N^0 <_{\mathfrak{K}} N^1$ in K_{λ^+} (so $N^0 \neq N^1$); such a pair exists as $K_{\lambda^{+2}} \neq \emptyset$. Let $N^\ell = \bigcup_{i<\lambda^+} N^\ell_i$ with $N^\ell_i \in K_\lambda$ being $\leq_{\mathfrak{K}}$-increasing continuously in i. Now

$E_0 = \{\delta < \lambda^+ : N^1_\delta \neq N^0_\delta$ and $N^1_\delta \cap N^0 = N^0_\delta\}$ is a club of λ^+. Without loss of generality $E_0 = \lambda^+$.

For each $c \in N^1 \backslash N^0$, the set

$$X_c =: \{i < \lambda^+ : c \in N^1_i \text{ and } (N^0_i, N^1_i, c) \text{ is minimal}\}$$

is empty or an end segment of λ^+, hence

$$\begin{aligned} E_1 = \{\delta < \lambda^+ : &\text{(i) } \delta \text{ limit and } N^1_\delta \nsubseteq N^0, \\ &\text{(ii) if } i < \delta \text{ and } p \in \mathcal{S}(N^0_i) \text{ is minimal inevitable} \\ &\quad \text{and realized in } N^0 \backslash N^0_\delta \text{ then it is} \\ &\quad \text{realized in } N^0_\delta \backslash N^0_i \text{ (actually automatic)}, \\ &\text{(iii) if } c \in N^1_\delta \backslash N^0 \text{ (hence } \exists i < \delta, c \in N^1_i) \text{ and } X_c \\ &\quad \text{is non-empty } \underline{\text{then}}\ \delta \in X_c \text{ and } \min(X_c) < \delta\} \end{aligned}$$

is a club of λ^+ (see 5.4).

Now for $\delta \in E_1$, we have $N^0_\delta <_{\mathfrak{K}} N^1_\delta$, so by 5.3(1) there is $c_\delta \in N^1_\delta \backslash N^0_\delta$ such that:

$$(N^0_\delta, N^1_\delta, c_\delta) \text{ is minimal,}$$
$$\text{tp}(c_\delta, N^0_\delta, N^1_\delta) \text{ is inevitable.}$$

As δ is limit, for some $i < \delta, c \in N^1_i$, also $\delta \in X_c$, hence there is j such that: $i < j < \delta$ & $j \in X_c$ hence (N^0_j, N^1_j, c) is minimal; choose such j_δ, c_δ. Let $\kappa = \text{cf}(\kappa) = \text{cf}(\alpha) \leq \lambda$, so for some j^*, c^* we have

$$S = \{\delta \in E_1 : \text{cf}(\delta) = \kappa, j_\delta = j^*, c_\delta = c^*\}$$

is stationary in λ^+.

Choose e closed $\subseteq E_1$ of order type $\alpha + 1$ with first element and last element in S; for $\zeta \in [j^*, \lambda^+)$ let $p_\zeta = \text{tp}(c^*, N^0_\zeta, N^1_\zeta)$. (In fact, we could have: all non-accumulation members of e are in S; no real help.)

Now $\langle N^0_\zeta, p_\zeta : \zeta \in e\rangle$ is as required (up to re-indexing)(clause (viii) holds by clause (ii) in the definition of E_1). $\blacksquare_{5.5}$

5.6 CLAIM: *Assume $\langle N_i, p_i : i \leq \alpha\rangle$ is as in 5.5, $\alpha < \lambda$ divisible by λ. <u>Then</u> any $p \in \mathcal{S}(N_0)$ is realized in N_α, moreover N_α is universal in K_λ over N_0.*

Proof: (Similar to the proof of 0.26; which is [Sh 300, II, §3]).

Let $N_0 \leq_{\mathfrak{K}} M_0 \in K_\lambda, a \in M_0 \backslash N_0$; we shall show that $\text{tp}(a, N_0, M_0)$ is realized in N_α.

Let $\alpha = \bigcup_{i<\alpha} S_i$, $\langle S_i : i < \alpha \rangle$ pairwise disjoint, each S_i unbounded in α, λ divides $\mathrm{otp}(S_i)$ and $\mathrm{Min}(S_i) \geq i$. We choose by induction on $i \leq \alpha$ the following:

$$N_i^1,\ M_i^1,\ h_i,\ \langle a_\zeta : \zeta \in S_i \rangle \quad \text{(the last one only if } i < \alpha\text{)}$$

such that:

(a) $N_i^1 \leq_{\mathfrak{K}} M_i^1$ are in K_λ,
(b) N_i^1 is $\leq_{\mathfrak{K}}$-increasing continuous in i,
(c) M_i^1 is $\leq_{\mathfrak{K}}$-increasing continuous in i,
(d) $(N_0^1, M_0^1) = (N_0, M_0)$,
(e) $\langle a_\zeta : \zeta \in S_i \rangle$ is a list of $\{c \in M_i^1 : c \text{ realizes } p_0\}$,
(f) h_i is an isomorphism from N_i onto N_i^1,
(g) $j < i \Rightarrow h_j \subseteq h_i$ and $h_0 = \mathrm{id}_{N_0}$,
(h) $a_i \in N_{i+1}^1$ (note: $M_i^1 \cap N_{i+1}^1 \neq N_i^1$ in general).

For $i = 0$: See clauses (d), (g),

$$N_0^1 = N_0, \quad M_0^1 = M_0, \quad h_0 = \mathrm{id}_{N_0}.$$

For $i =$ limit: Let $N_i^1 = \bigcup_{j<i} N_j^1$ and $M_i^1 = \bigcup_{j<i} M_j^1$ and $h_i = \bigcup_{j<i} h_j$ and lastly choose $\langle a_\zeta : \zeta \in S_i \rangle$ by clause (e).

<u>For $i = j+1$</u>: Note a_j is already defined; it belongs to M_j^1 and it realizes p_0.

<u>Case 1</u>: $a_j \in N_j^1$ (so clause (h) is no problem).

Use amalgamation on N_j, N_i, M_j^1 and the mapping id_{N_j}, h_i, i.e.

$$\begin{array}{ccc} N_i & \longrightarrow & M_i^1 \\ {\scriptstyle id_{N_j}} \uparrow & & \uparrow {\scriptstyle id_{N_0^1}} \\ N_j & \xrightarrow{\ h_i\ } & N_j^1 \end{array}$$

<u>Case 2</u>: $a_j \notin N_j^1$.

Then $\mathrm{tp}(a_j, N_j^1, M_j^1)$ is not algebraic, extending the minimal type $p_0 \in \mathcal{S}(N_0)$. Also by clause (viii) of 5.5 there is $c \in N_i \setminus N_j$ which realizes p_0. As $p_0 \in \mathcal{S}(N)$ is minimal

$$h_j\left(\mathrm{tp}(c, N_j, N_i)\right) = \mathrm{tp}(a_j, N_j^1, M_j^1),$$

so acting as in Case 1 we can also guarantee $h_i(c) = a_j$, so $a_j \in \mathrm{Rang}(h_i) = N_i^1$ as required.

In the end we have $N_\alpha^1 \leq_{\mathfrak{K}} M_\alpha^1$. If $N_\alpha^1 = M_\alpha^1$, then $h_\alpha^{-1} \restriction M_0 = h_\alpha^{-1} \restriction M_0^1 = h_\alpha^{-1} \restriction N_0^1$ show that M_0 can be embedded into N_α over N_0 as required. So assume

$N^1_\alpha <_{\mathfrak{K}} M^1_\alpha$. Now $p_\alpha \in \mathcal{S}(N_\alpha)$ is inevitable hence $h_\alpha(p_\alpha) \in \mathcal{S}(N^1_\alpha)$ is inevitable. Hence some $d \in M^1_\alpha \backslash N^1_\alpha$ realizes $h_\alpha(p_\alpha)$, hence d realizes $h_\alpha(p_\alpha) \restriction N^1_0 = p_0$; also α is a limit ordinal, so for some $i < \alpha$, $d \in M^1_i$, hence for some $\zeta \in S_i$ we have $a_\zeta = d$, hence

$$d = a_\zeta \in N^1_{\zeta+1} \subseteq N^1_\alpha,$$

contradicting the choice of d. So we are done. $\blacksquare_{5.6}$

5.7 Conclusion: If $N \in K_\lambda$ then:

(a) $|\mathcal{S}(N)| = \lambda$,

(b) there is $N_1, N <_{\mathfrak{K}} N_1 \in K_\lambda$ such that N_1 is universal over N in K_λ,

(c) for any regular $\kappa \leq \lambda$ we can demand that $(N_1, c)_{c \in N}$ is (λ, κ)-saturated (see 0.28(1)).

5.8 Remark: In fact amalgamation in λ and stability in λ (i.e. (a) of 5.7) implies (b) and (c) of 5.7.

5.9 Conclusion: The $N \in K_{\lambda^+}$ is saturated above λ (i.e. over models in K_λ!).

5.10 Claim: *Assume $\kappa = cf(\kappa) \leq \lambda$. There are $N_0, N_1, a, N_0^+, N_1^+$ such that*

(i) *$(N_0, N_1, a) \in K^3_\lambda$ and*

(ii) *$(N_0, N_1, a) \leq (N_0^+, N_1^+, a) \in K^3_\lambda$ and*

(iii) *N_0^+ is (λ, κ)-saturated over N_0,*

(iv) *$tp(a, N_0, N_1)$ is minimal inevitable and*

(v) *$tp(a, N_0^+, N_1^+)$ is minimal inevitable.*

Proof: As in the proof of 5.5 because

$$E_2 = \{\delta : \text{ for every } i < \delta, N^0_\delta \text{ is saturated over} N_i \text{ of cofinality } \mathrm{cf}(\delta)\}$$

is a club of λ^+. $\blacksquare_{5.10}$

5.11 Claim: (1) *In K_λ we have disjoint amalgamation.*

(2) *If $M \leq_{\mathfrak{K}} N$ are in K_λ and $p \in \mathcal{S}(M)$ non-algebraic <u>then</u> for some N', c we have: $N \leq_{\mathfrak{K}} N' \in K_\lambda$ and $c \in N' \backslash N$ realizes p.*

Proof: (1) First note:

⊗ if $M \leq_{\mathfrak{K}} N$ in K_λ we can find $\alpha < \lambda^+$, and sequence $\langle M_i : i \leq \alpha \rangle$ which is $\leq_{\mathfrak{K}}$-increasing continuous, and $\langle a_i : i < \alpha \rangle$ such that (M_i, M_{i+1}, a_i) is minimal and reduced and $N \leq_{\mathfrak{K}} M_\alpha, M = M_0$.

[Why? There is a minimal reduced pair, hence we can find $\langle M_i : i < \lambda^+ \rangle \leq_{\mathfrak{K}}$-increasing continuous, (M_i, M_{i+1}, a_i) minimal reduced and $M = M_0$. So by 5.9

we know $\bigcup_{i<\lambda^+} M_i \in K_{\lambda^+}$ is saturated, hence we can embed N into $\bigcup_{i<\lambda^+} M_i$ over N so this embedding is into some $M_\alpha, \alpha < \lambda^+$.]

Therefore given $M \leq_{\mathfrak{K}} M^1, M^2$, without loss of generality $M^\ell = M^\ell_{\alpha_\ell}$, $\langle (M^\ell_i, a^\ell_i) : i \leq \alpha_\ell \rangle$ as above, and start to amalgamate using the extension property and "reduced".

(2) Follows from part (1). $\blacksquare_{5.11}$

5.12 Remark: We could prove 5.11 earlier using "reduced triples". I.e. note that for some $\langle M^1_i : i < \lambda^+ \rangle \in \mathbf{Seq}_{\lambda^+}[\mathbf{C}^1_{\mathfrak{K},\lambda^+}]$, for each i for some a the triple $(M^1_i, M^1_{i+1}, a) \in K^3_\lambda$ is reduced. Hence if $M \leq_{\mathfrak{K}} N$ from K_λ, for some $\bar{M} = \langle M_i : i \leq \alpha \rangle$, $\leq_{\mathfrak{K}}$-increasing continuous, $(M_i, M_{i+1}, b_i) \in K^3_\lambda$ is reduced, $M_0 = M, N \leq M_\alpha \in K_\lambda$ (otherwise find $\langle M^2_i : i < \lambda^+ \rangle \in \mathbf{Seq}_{\lambda^+}[\mathbf{C}^1_{\mathfrak{K},\lambda^+}]$ with $(M_i, M_{i+1}) \cong (M, N)$, hence $M^1 = \bigcup_{i<\lambda^+} M^1_i, M^2 = \bigcup_{i<\lambda^+} M^2_i$ are non-isomorphic members of K_{λ^+}, contradiction). Now prove by induction on $\beta \leq \alpha$ that if $M \leq_{\mathfrak{K}} N_0 \in K_\lambda$ then N_0, M_β has disjoint amalgamation over $M_0 = M$ (i.e. we need to decompose only one side).

5.13 Question: If $M \in K_\lambda, p \in \mathcal{S}(M)$ is minimal, is it reduced? Or at least, if $M_0 \leq_{\mathfrak{K}} M_1$ are in $K_\lambda, p_1 \in \mathcal{S}(M_\ell)$ is non-algebraic, $p_0 = p_1 \restriction M_0, p_0$ is minimal and reduced, is also p_1 reduced?

It is probably true and would somewhat simplify our work, but we have to go around it fulfilling our aims (here and in [Sh 600]). Now 5.5 is an approximation. It can be proved if $\lambda < \lambda^{\aleph_0}$ or there are E.M. models.

6. A proof for $\mathfrak{K}$ categorical in λ^{+2}

6.1 Hypothesis: Assume the model theoretic assumptions from 4.1 + 5.1, and so the further model theoretic properties deduced in §4 + §5. We use 4.8 heavily.

6.2 Definition: (1) We say $(M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda$ (has **unique** (disjoint) **amalgamation** in K_λ) when

(a) $M_0, M_1, M_2 \in K_\lambda$;

(b) $M_0 \leq_{\mathfrak{K}} M_1$ and $M_0 \leq_{\mathfrak{K}} M_2$;

(c) if for $i = 1, 2$ we have $g^i_\ell \colon M_\ell \to N_i \in K_\lambda$ such that:

  (i) g^i_ℓ a $\leq_{\mathfrak{K}}$-embedding,

  (ii) $g^i_0 \subseteq g^i_1$ and $g^i_0 \subseteq g^i_2$ for $i = 1, 2$,

  (iii) $\mathrm{Rang}(g^i_1) \cap \mathrm{Rang}(g^i_2) = \mathrm{Rang}(g^i_0)$ (disjoint amalgamation) for $i = 1, 2$,

then we can find $N \in K_\lambda$ and $\leq_{\mathfrak{K}}$-embeddings

$$f^i \colon N_i \to N \quad \text{for } i = 1, 2$$

such that

$$\bigwedge_{\ell<3} f^1 \circ g^1_\ell = f^2 \circ g^2_\ell.$$

(2) Let $K^{2,\mathrm{uq}}_\lambda$ be the class of pairs (M_0, M_2) such that $M_0 \leq_{\mathfrak{K}} M_2$ are both in K_λ and $[M_0 \leq_{\mathfrak{K}} M_1 \in K_\lambda \Rightarrow (M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda]$ and let $K^{3,\mathrm{uq}}_\lambda$ be the class of pairs $(M_0, M_2) \in K^{2,\mathrm{uq}}_\lambda$ satisfying $M_0 \neq M_2$.

6.3 CLAIM: (1) *If* $(M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda$ *then*

(a) $(M_0, M_2, M_1) \in K^{\mathrm{uniq}}_\lambda$,

(b) *if* $M_0 \leq_{\mathfrak{K}} M_2' \leq_{\mathfrak{K}} M_2$ *then* $(M_0, M_1, M_2') \in K^{\mathrm{uniq}}_\lambda$.

(2) *Assume* $M_0 \leq_{\mathfrak{K}} M_2$ *are from* K_λ *and* $M_1 \in K_\lambda$ *is universal over* M_0. *Then* $(M_0, M_2) \in K^{2,\mathrm{uq}}_\lambda \Leftrightarrow (M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda$.

Proof: (1)(a) Trivial.

(b) Chase arrows (using disjoint amalgamation, i.e. 5.11).

(2) Follows by 6.3(1)(a)+(b) and the definition. $\blacksquare_{6.3}$

6.4 LEMMA: *Suppose*

⊗ *there is* $(M_0, M_1, M_2) \in K^{\mathrm{uniq}}_\lambda$ *such that* $M_0 \neq M_2$ *and* M_1 *is universal over* M_0.

Then: there are $N^0 <_{\mathfrak{K}} N^1$ *in* K_{λ^+} *such that:*

(a) $N^0 \neq N^1$,

(b) *for every* $c \in N^1 \backslash N^0$ *there is* $M = M_c$ *satisfying* $N^0 \leq_{\mathfrak{K}} M \leq_{\mathfrak{K}} N^1$ *and* $N^0 \neq M$ *and* $c \in N^1 \backslash M$.

Proof: Choose $\langle N^0_i : i < \lambda^+ \rangle$, a sequence of members of K_λ which is $\leq_{\mathfrak{K}}$-increasing continuous, such that:

$$(N^0_i, N^0_{i+1}) \cong (M_0, M_2).$$

So $N^0_i \neq N^0_{i+1}$, hence $N_0 = \bigcup_{i<\lambda^+} N^0_i \in K_{\lambda^+}$, and without loss of generality $|N_0| = \lambda^+$.

We now choose, by induction on $i < \lambda^+, N^1_i$ and $M_{i,c}$ for $c \in N^1_i \backslash N^0_i$ such that:

(a) $N^0_i \leq_{\mathfrak{K}} N^1_i \in K_\lambda$ and $N^0_i \neq N^1_i$,

(b) N^1_i is $\leq_{\mathfrak{K}}$-increasing continuous in i,

(c) $j < i \Rightarrow N^1_j \cap N^0_i = N^0_j$, moreover $N^1_i \cap |N_0| = N^0_i$,

(d) $N^0_i \leq_{\mathfrak{K}} M_{i,c} \leq_{\mathfrak{K}} N^1_i$,

(e) $c \notin M_{i,c}$,

(f) $N^0_i \neq M_{i,c}$,

(g) if $j < i$ and $c \in N_j^1 \backslash N_j^0$ then $M_{i,c} \cap N_j^1 = M_{j,c}$.

For $i = 0$: Choose N_i^1 such that $N_i^0 \leq_{\mathfrak{K}} N_i^1$, $(N_i^1, c)_{c \in N_i^0}$ saturated (any cofinality will do); then by disjoint amalgamation it is easy to define the $M_{0,c}$ (remembering clause (c)).

For i limit: Straightforward.

For $i = j+1$: First we disjointly amalgamate getting $N_i' \in K_\lambda$ such that $N_i^0 \leq N_i', N_j^1 \leq_{\mathfrak{K}} N_i'$ and $|N_i'| \cap |N_0| = |N_i^0|$ (as set of elements). Let N_i^1 be such that:

$$N_i' \leq_{\mathfrak{K}} N_i^1 \in K_\lambda,$$
$$(N_i^1, c)_{c \in N_i'} \text{ is saturated (any cofinality will do),}$$
$$|N_i^1| \cap |N_0| = |N_i^0|.$$

Lastly, we shall find the $M_{i,c}$'s; the point is that $(N_j^0, N_i^0, N_j^1) \in K_\lambda^{\text{uniq}}$ (by 6.3(2)).

By $\otimes$ and Claim 6.3 we could have done the amalgamation in two steps and use uniqueness. Then by uniqueness of saturated extensions embed the result inside N_i^1 and similarly deal with new c's.

Now let $N_1 =: \bigcup_{1<\lambda^+} N_i^1$ and for $c \in N_1 \backslash N_0$ let $M_c = \bigcup\{M_{i,c} : c \in N_i'\}$; they are as required. $\blacksquare_{6.4}$

Remark: The proof of 6.5 below is like [Sh 88, proof of 2.8 stage (c)]. The aim is to contradict that under $I(\lambda^{+3}, \mathfrak{K}) = 0$ there are maximal triples.

6.5 Conclusion: Assume $\mathfrak{K}$ has amalgamation in λ^+. With $\otimes$ of 6.4, then there is no maximal triple (M, N, a) in $K^3_{\lambda^+}$.

Proof: We can get by 6.4 a contradiction.

[Why? Assume $(N_0, N_2, a) \in K^3_{\lambda^+}$ maximal, (N^0, N^1) as in the conclusion (i.e. (a) + (b)) of 6.4; by categoricity in λ^+ without loss of generality $N_0 = N^0$ and let $N_1 = N^1$. Now $\mathfrak{K}$ has amalgamation for λ^+, so there are $N \in K_{\lambda^+}$ and f such that $f\colon N_2 \to N$ is a $\leq_{\mathfrak{K}}$-embedding of N_2 into N over N_0 and $N_1 \leq_{\mathfrak{K}} N$. If $f(a) \notin N_1$, then $(N_0, N_2, a) <_f (N_1, N, f(a))$ contradict maximality. If $f(a) \in N_1$, then $M_{f(a)}$ is well defined (see 6.4) and $(N_0, N_2, a) <_f (M_{f(a)}, N, f(a))$ contradicts maximality.] $\blacksquare_{6.5}$

6.6 Remark: (1) Another proof is to replace the assumption "$\mathfrak{K}$ has amalgamation in λ^+" by "$I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$". We start with N_0, N_1, N_2, a as above and build, for every $S \subseteq \lambda^{+2}$, a sequence $\langle M_\alpha^S : \alpha < \lambda^{+2}\rangle$ of members of K_{λ^+}, which is $\leq_{\mathfrak{K}}$-increasing continuous, and $\alpha \in S \Rightarrow (M_\alpha^S, M_{\alpha+1}^S, a_\alpha^S) \cong (N_0, N_2, a)$, and $\alpha \in \lambda^{+2} \backslash S \Rightarrow (M_\alpha^S, M_{\alpha+1}^S) \cong (N_0, N_1)$ which are as in (a)+(b) of 6.4. Let

$M^S = \bigcup_{\alpha<\lambda^{+2}} M^S_\alpha \in K_{\lambda^{+2}}$ and from $M^S/\cong$ we can reconstruct $S/\mathcal{D}_{\lambda^{+2}}$. So here we use $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ but no need for the definitional weak diamond for λ^{++}.

(2) Note that if $2^{\lambda^+} < 2^{\lambda^{++}}$ then the assumption of 6.6(1) implies the assumption of 6.5.

6.7 Claim: *Assume*

(*) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ *(or at least the definitional weak diamond for* λ^+, λ^{++}*) and*

(**) (α) WDmId(λ^+) *is not a* λ^{++}*-saturated ideal, or*
$(\beta) K_{\lambda^{+3}} = \emptyset$ *and* $\neg(\lambda^+ = 2^\lambda \geq \beth_\omega)$.

If $\bigotimes$ *of* 6.4 *fails, we get* $I(\lambda^{+2}, K) \geq 2^{\lambda^{+2}}$.

Proof: First, if (**)(α) holds then by 3.19 and 6.8 below we get the conclusion. Second, if $K_{\lambda^{+3}} = \emptyset$ and $2^{\lambda^+} > \lambda^{++}$ then there is $M_2^* \in K_{\lambda^{++}}$ which is $\leq_{\mathfrak{K}}$-maximal hence saturated (above λ^+ and above λ as $\mathfrak{K}_{\lambda^+}$ and $\mathfrak{K}_\lambda$ have amalgamation), and let $M_1 \leq_{\mathfrak{K}} M_2$, $M_1 \in K_{\lambda^+}$; now by the proof of 6.7 as $\lambda \notin \text{WDmId}_\mu(\lambda^+)$ for $\mu = \lambda^{+3}$ by 1.2(2) second case (as in 3.23) there is a $\leq_{\mathfrak{K}}$-extension of M_1 in K_{λ^+} not $\leq_{\mathfrak{K}}$-embeddable into M_2. Third, if $2^{\lambda^+} = \lambda^{++}$ then necessarily $\lambda < 2^\lambda < 2^{\lambda^+} = \lambda^{++}$ so $2^\lambda = \lambda^+$, so by [Sh 460], if $\lambda \geq \beth_\omega$ then $\diamond_{\lambda^+}$ hence WDmId(λ^+) is not λ^{++}-saturated, a case we have dealt with. Together we are done. $\blacksquare_{6.7}$

The following serves to prove 6.7.

6.8 Claim: *Assume* $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$.

If $K_\lambda^{3,\text{uq}} = \emptyset$ *(see Definition 6.2(2)),* *<u>then</u> there is an amalgamation choice function* F *for* $\mathbf{C} = \mathbf{C}^0_{\mathfrak{K},\lambda^+}$, *with the weak* λ^+*-coding property.*

Proof: The point is that if $\bar{M} = \langle M_\alpha : \alpha < \lambda^+\rangle \in \mathbf{Seq}_{\lambda^+}[\mathbf{C}]$ and $\alpha < \lambda^+, M_\alpha <_{\mathfrak{K}} N \in K_\lambda$, <u>then</u> for some $\beta \in (\alpha, \lambda^+)$ we have:

M_β is universal over M_α, so as $K_\lambda^{3,\text{uq}} = \emptyset$ necessarily $(M_\alpha, M_\beta, N) \notin K_\lambda^{\text{uniq}}$ and the rest should be clear.

Of course, we use the extension property. $\blacksquare_{6.8}$

6.9 Remark: We can work in the context of §3; we need the existence of a saturated (equivalently super limit) $M \in K_{\lambda^+}$. We now say how to replace $\mu_{\text{wd}}(\lambda^{+2})$ by $2^{\lambda^{+2}}$.

6.10 CLAIM: (1) *Assume each $M \in K_{\lambda^+}$ is saturated above λ.*

If $(M, N, a) \in K^3_\lambda$, it and every $(M', N', a) \in K^3_\lambda$ above it has the extension property, but for every $(M'', N'', a) \geq (N, N, a)$ (all in K^3_λ) for some $M^* \geq_{\mathfrak{K}} M''$ from K_λ, in amalgamation $(M^*, N^*, a) \geq (M'', N'', a)$ the type of $M^* \cup N$ inside N^* is not determined, then some F (actually $\mathbf{F}^*$) has the λ^+-coding property.
(2) *If above we just require that the type of $M^* \cup N''$ inside N^* is not determined, then some F (actually $\mathbf{F}^*$) has weak λ^+-coding.*
(3) *We can restrict ourselves to disjoint embedding.*

6.11 DISCUSSION: We get $IE(\lambda^{+2}, \mathfrak{K}) = 2^{\lambda^{+2}}$ when $(2^{\lambda^+})^+ < 2^{\lambda^{+2}}$. See more in [Sh 600].

We now prove 0.2.

6.12 THEOREM: *Assume* $(*)$ *of* 6.7 *(or at least the conclusion of* 6.7*). Then* $I(\lambda^{+2}, K) = 1 \Rightarrow I(\lambda^{+3}, K) > 0$.

Remark: As in [Sh 88, §3].

Proof: By 0.20(1) it is enough to show that for some $M \in K_{\lambda^{++}}$ there is M', $M \leq_{\mathfrak{K}} M' \in K_{\lambda^{++}}, M \neq M'$. [Why? As then we can choose by induction on $i < \lambda^{+3}$ models $M_i \in K_{\lambda^{+2}}$, $\leq_{\mathfrak{K}}$-increasing continuous, $M_i \neq M_{i+1}$, for $i = 0$ use $K_{\lambda^{+2}} \neq \emptyset$, for i limit take union, for $i = j + 1$ use the previous sentence; so $M_{\lambda^{+3}} = \bigcup\{M_i : i < \lambda^{+3}\} \in K_{\lambda^{+3}}$ as required.]

By 6.7, the statement $\bigotimes$ of 6.4 holds so we can find (N^0, N^1) as there, hence by 6.5 there is in $K^3_{\lambda^+}$ no maximal member. This implies (easy, see 2.6(6)) that there are $M^* \leq_{\mathfrak{K}} N^*$ from $K_{\lambda^{+2}}$ such that $M^* \neq N^*$ which, as mentioned above (by categoricity in λ^+), suffices. $\blacksquare_{6.12}$

7. Extensions and conjugacy

7.1 HYPOTHESIS: Assume the model theoretic assumptions from 4.1+5.1 and the further model theoretic properties deduced since then (but not in 6.7, 6.12), or just

(a) $\mathfrak{K}$ is an abstract elementary class,
(b) $\mathfrak{K}$ has amalgamation in λ,
(c) $\mathfrak{K}$ is categorical in λ (can be weakened),
(d) $\mathfrak{K}$ is stable in λ (see 5.7, clause (a)),
(e) there is an inevitable $p \in \mathcal{S}(N)$ for $N \in K_\lambda$ (holds by 5.3),
(f) the basic properties in type theory.

We now continue toward eliminating the use of $I(\lambda^{++}, K) = 1$ (in 6.12), and give more information. We first deal with the nice types in $\mathcal{S}(N), N \in K_\lambda$, in particular the realize/materialize problem which is here: if $N_1 \leq_{\mathfrak{K}} N_2$ are in $K_\lambda, p_\ell \in \mathcal{S}(N_\ell)$ is minimal, $p_1 \leq p_2$, are they conjugate? (i.e. does $p_2 \in \mathcal{S}_{p_1}(N_2)$?).

7.2 CLAIM: *If $N \in K_\lambda$ and $p \in \mathcal{S}(N)$ is minimal and reduced or just p is reduced (see Definition 2.3(7)), <u>then</u> p is inevitable.*

Proof: Suppose N, p form a counterexample. We can then find N_1 and a such that $N \leq_{\mathfrak{K}} N_1 \in K_\lambda, a \in N_1 \backslash N$ and $p = \text{tp}(a, N, N_1)$ and (N, N_1, a) is reduced. As p is not inevitable, there is N_2 such that: $N \leq_{\mathfrak{K}} N_2 \in K_\lambda, N \neq N_2$ but no element of N_2 realizes p. By amalgamation in K_λ, without loss of generality there is $N_3 \in K_\lambda$ such that $\ell \in \{1,2\} \Rightarrow N_\ell \leq_{\mathfrak{K}} N_3$. By 5.3 (i.e. 7.1(e)) there is $q \in \mathcal{S}(N)$, which is inevitable so there are $c_\ell \in N_\ell$ with $q = \text{tp}(c_\ell, N, N_\ell)$ for $\ell \in \{1,2\}$. By the equality of types (and amalgamation in K_λ) there is $N^+ \in K$, a $\leq_{\mathfrak{K}}$-extension of N_1 and a $\leq_{\mathfrak{K}}$-embedding f of N_2 into N^+ over N such that $f(c_2) = c_1$; so without loss of generality $N^+ = N_3$ and f is the identity, hence $c_1 = c_2$. Now $a \notin N_2$ as $p = \text{tp}(a, N, N_1)$ is not realized in N_2. So $(N, N_1, a) \leq (N_2, N_3, a)$ and $N_2 \cap N_1 \backslash N \neq \emptyset$, contradicting "$(N, N_1, a)$ is reduced". $\blacksquare_{7.2}$

7.3 CLAIM: (1) *If $\kappa = cf(\kappa) \leq \lambda$ and $\bar{N} = \langle N_i : i \leq \omega\kappa \rangle$ is an $\leq_{\mathfrak{K}}$-increasingly continuous sequence, $N_i \in K_\lambda, N_{i+1}$ universal over N_i, and $p \in \mathcal{S}(N_{\omega\kappa})$ is minimal reduced (or minimal inevitable) <u>then</u> for some $i < \omega\kappa$ we have $p \restriction N_i \in \mathcal{S}(N_i)$ is minimal (so p is the unique, non-algebraic extension of $p \restriction N_i$ in $\mathcal{S}(N_{\omega\kappa})$ (and, of course, there is one)).*

(2) *If $\lambda \geq \kappa = cf(\kappa), \bar{N} = \langle N_i : i \leq \kappa \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous in K_λ and $p \in \mathcal{S}(N_\kappa)$ is minimal and reduced and the set $Y =: \{i < \kappa : N_{i+1}$ is (λ, κ)-saturated over $N_i\}$ is unbounded in κ <u>then</u> for every large enough $i \in Y$ there is an isomorphism f from N_{i+1} onto N_κ which is the identity on N_i and*

(*) *f maps $p \restriction N_{i+1} \in \mathcal{S}(N_{i+1})$ to $p \in \mathcal{S}(N_\kappa)$.*

Hence as p is minimal reduced, so is $p \restriction N_{i+1}$.

Proof: (1) We can choose $(N_i^0, N_i^1, a) \in K_\lambda^3$ for $i < \lambda^+$ reduced, $\leq$-increasing continuous such that $N_i^0 \neq N_{i+1}^0$. Let $N_\ell = \bigcup_{i<\lambda^+} N_i^\ell$. As in the proof of 5.5 for $c \in N_1 \backslash N_0$,

$$I_c^* = \{j < \lambda^+ : c \in N_j^1 \text{ and } \text{tp}(c, N_j^0, N_j^1) \text{ is minimal}\}$$

is empty or is an end segment of λ^+ and

$$E = \{\delta < \lambda^+ : \underline{\text{if}}\ c \in N^1_\delta \text{ and } I^*_c \neq \emptyset\ \underline{\text{then}}\ I^*_c \cap \delta \text{ is an unbounded subset of } \delta; \text{and if } \alpha < \delta \text{ then, for some } \beta \in (\alpha, \delta), N^0_\beta \text{ is universal over } N^0_\alpha \text{ and if } Pr \text{ is one of the properties reduced and/or inevitable and/or minimal and there is } i \geq \delta \text{ such that } (N^0_i, N^1_i, c) \text{ has } Pr, \text{ then there are arbitrarily large such } i < \delta\}$$

is a club of λ^+; for the universality demand in the definition of E use categoricity in λ^+. Let $\delta \in \operatorname{acc}(\operatorname{acc}(E)), \operatorname{cf}(\delta) = \kappa$, let $\langle \alpha_\zeta : \zeta < \omega\kappa \rangle$ be an increasing continuous sequence of ordinals from E with limit δ, now set $\alpha_{\omega\kappa} = \delta$ and $N'_\zeta =: N^0_{\alpha_\zeta}$.

So there is an isomorphism f from $N_{\omega\kappa}$ onto $N^0_{\alpha_{\omega\kappa}}$ such that for every $\zeta < \omega\kappa$ we have $N^0_{\alpha_{2\zeta}} \leq_{\mathfrak{K}} f(N_{\alpha_{2\zeta}}) \leq_{\mathfrak{K}} N^0_{\alpha_{2\zeta+1}}$ (so if ζ is a limit ordinal, then $N^0_{\alpha_\zeta} = N^0_{\alpha_{2\zeta}} = f(N_\zeta)$), so without loss of generality f is the identity. As $p \in \mathcal{S}(N_{\omega\kappa})$ is inevitable (by assumption or by 7.2) and $N_{\omega\kappa} = N^0_{\alpha_{\omega\kappa}} <_{\mathfrak{K}} N^1_{\alpha_{\omega\kappa}}$, for some $c \in N^1_{\alpha_{\omega\kappa}} \setminus N^0_{\alpha_{\omega\kappa}}$ we have $p = \operatorname{tp}(c, N^0_{\alpha_{\omega\kappa}}, N^1_{\alpha_{\omega\kappa}})$, so for some $\beta < \alpha_{\omega\kappa}$ we have $c \in N^1_\beta$. As p is minimal (by assumption) clearly $\delta \in I_c$, but $\delta \in E$ so $\operatorname{Min}(I_c) < \delta$; but I_c is an end segment of λ^+, hence without loss of generality for some $\zeta < \omega\kappa$ we have $\beta = \alpha_\zeta \in I_c$. So for $\xi \in (\zeta, \omega\kappa)$, both $p \in \mathcal{S}(N_{\omega\kappa})$ and $p \restriction N_\xi \in \mathcal{S}(N_\xi)$ are non-algebraic extensions of the minimal $p \restriction N^0_{\alpha_\zeta} \in \mathcal{S}(N^0_{\alpha_\zeta})$ and $N^0_{\alpha_\zeta} \leq_{\mathfrak{K}} N_\xi \leq_{\mathfrak{K}} N_{\omega\kappa}$, all in K_λ, so we have proved part (1).
(2) Without loss of generality every $\zeta < \kappa$ is in Y. We can find $\langle N'_\zeta : \zeta \leq \kappa\kappa \rangle$ as in part (1), moreover satisfying "$N'_{\zeta+1}$ is (λ, κ)-saturated over N_ζ" and such that: for every $\zeta \leq \kappa$ we have $N_\zeta = N'_{\kappa\zeta}$. So again choose $\zeta < \kappa$ as there; we set $\beta = \alpha_{\kappa\zeta} \in I^*_c$. If $\xi \in Y$ and $\kappa\xi > \zeta$, clearly by the uniqueness of (λ, κ)-saturated models there is an isomorphism f from $N_{\xi+1} = N'_{\kappa(\xi+1)}$ onto $N_\kappa = N'_{\kappa\kappa}$ over $N_\xi = N'_{\kappa\kappa}$, and $f(p \restriction N_{\xi+1}) = p$ is proved as above by the uniqueness of the non-algebraic extension. $\blacksquare_{7.3}$

7.4 CLAIM: (1) *If $M_0 \leq_{\mathfrak{K}} M_1$ are in K_λ and the types $p_\ell \in \mathcal{S}(M_\ell)$ are minimal reduced, for $\ell = 0, 1$ and $p_0 = p_1 \restriction M_0$ <u>then</u> p_0, p_1 are conjugate (i.e. there is an isomorphism f from M_0 onto M_1 such that $f(p_0) = p_1$).*
(2) If in addition $M \leq_{\mathfrak{K}} M_0$ and M_0, M_1 are (λ, κ)-saturated over M, <u>then</u> p_0, p_1 are conjugate over M.

Remark: Note that p minimal (or reduced) implies that p is not algebraic.

Proof: (1) Let $\langle (N_i^0, N_i^1, a) : i < \lambda^+ \rangle$ and E be as in the proof of 7.3 and $\kappa = \mathrm{cf}(\kappa) \leq \lambda$. For each $\delta \in S_\kappa =: \{\alpha < \lambda^+ : \alpha \in E \text{ and } \mathrm{cf}(\alpha) = \kappa\}$, and minimal reduced $p \in \mathcal{S}(N_\delta^0)$, we know that for some $i_p < \delta, p \restriction N_{i_p}^0$ is minimal reduced [why? by 7.3(1),(2)] and some $q_p \in \mathcal{S}(N_{i_p}^0)$ is conjugate to p say by g_p an isomorphism from N_δ^0 onto $N_{i_p}^0$. For $\kappa = \mathrm{cf}(\kappa) \leq \lambda, q \in \mathcal{S}(N_i^0), i < \lambda^+, r \in \mathcal{S}(N_i^0)$ minimal let

$$\begin{aligned} A_{q,r}^{\kappa,i} = \{\delta < \lambda^+ : {} & \text{there is a type } p \text{ such that } r \subseteq p \in \mathcal{S}(N_\delta^0), p \text{ non-algebraic} \\ & (\text{this determines } p), p \text{ minimal reduced}, i_p = i, q_p = q \\ & (\text{and clearly } p \restriction N_i^0 = r) \text{ and } \mathrm{cf}(\delta) = \kappa\}. \end{aligned}$$

Next let

$$\begin{aligned} E_1 = \{\delta < \lambda^+ : {} & \text{for every } \kappa = \mathrm{cf}(\kappa) \leq \lambda, \\ & r, q \in \mathcal{S}(N_i^0) \text{ and } i < \delta, \underline{\text{if}}\ A_{q,r}^{\kappa,i} \text{ is well defined and} \\ & \text{unbounded in } \lambda^+ \ \underline{\text{then}} \text{ it is unbounded in } \delta\}. \end{aligned}$$

So if $\delta_1 \in E_1, \kappa = \mathrm{cf}(\delta_1), p_1 \in \mathcal{S}(N_{\delta_1}^0)$ is minimal reduced, then we can find $\delta_0 < \delta_1, \mathrm{cf}(\delta_0) = \kappa$, and $p_0 \in \mathcal{S}(N_{\delta_0}^0)$ minimal reduced with $q_{p_1} = q_{p_0}, i_{p_1} = i_{p_0}, p_0 \restriction N_{i_{p_0}}^0 = p_1 \restriction N_{i_{p_1}}^0$; call it r, it is necessarily minimal.

As p_1, p_0 extend $r, N_{i_{p_0}}^0 = N_{i_{p_1}}^0 \leq_{\mathfrak{K}} N_{\delta_0}^0 \leq_{\mathfrak{K}} N_{\delta_1}^0$, necessarily $p_1 = p_0 \restriction N_{\delta_0}^0$, and also they are both conjugate to $q_{p_0} = q_{p_1}$, hence they are conjugate.

Next we prove

(∗) if $M_0 <_{\mathfrak{K}} M_1$ are in K_λ, M_1 is (λ, κ)-saturated over M_0, $p_0' \in \mathcal{S}(M_0)$ is minimal reduced and $p_0' \leq p_1' \in \mathcal{S}(M_1), p_1'$ non-algebraic, then p_0', p_1' are conjugate.

Above we have a good amount of free choice in choosing $p_1 \in \mathcal{S}(N_{\delta_1}^0)$ (it should be minimal and reduced) so we could have chosen p_1 to be conjugate to p_0', i.e. in $\mathcal{S}_{p_0'}(N_{\delta_1}^0)$; now also the corresponding p_0 is conjugate to p_1, hence p_0 is conjugate to p_0', so we can find an isomorphism f_0 from M_0 onto $N_{\delta_0}^0, f_0(p_0') = p_0$, and extend it to an isomorphism f_1 from M_1 onto $N_{\delta_1}^0$, so necessarily $f_1(p_1') = p_1$ (as p_1 is the unique non-algebraic extension of p_0 in $\mathcal{S}(M_{\delta_1})$). As p_0, p_1 are conjugate through $(g_{p_1})^{-2} \circ g_{p_0}$, also p_0', p_1' are conjugate. So (∗) holds.

Now assume just $M_0 \leq_{\mathfrak{K}} M_1$ are in $K_\lambda, p_0 \in \mathcal{S}(M_0)$ minimal reduced, $p_1 \in \mathcal{S}(M_1)$ the unique non-algebraic extension of p_0 and it is reduced (and necessarily minimal). There is $M_2, M_1 \leq_{\mathfrak{K}} M_2 \in K_\lambda, M_2$ is (λ, κ)-saturated over M_1, hence also over M_0, and let p_2 be the unique non-algebraic extension of p_1 in $\mathcal{S}(M_2)$; hence p_2 is also the unique non-algebraic extension of p_0 in $\mathcal{S}(M_2)$.

Using $(*)$ on (M_0, M_2, p_0, p_2) and on (M_1, M_2, p_1, p_2) we get that p_0, p_2 are conjugate and that p_1, p_2 are conjugate resp., hence p_1, p_2 are conjugate, the required result.

(2) A similar proof. $\blacksquare_{7.4}$

7.5 CLAIM: (1) *Assume $M_1 \leq_{\mathfrak{K}} M_2$ are in K_λ and M_2 is (λ, κ)-saturated over M_1. If $p_1 \in \mathcal{S}(M_1)$ is minimal and reduced, <u>then</u> p_2, the unique non-algebraic extension of p_1 in $\mathcal{S}(M_2)$, is reduced (and, of course, minimal).*

(2) *There is no need to assume "p_1 reduced".*

Proof: (1) We can find $\langle N_i : i \leq \kappa \rangle$, an $\leq_{\mathfrak{K}}$-increasingly continuous sequence in K_λ such that N_{i+1} is (λ, κ)-saturated over N_i and $N_\kappa = M_1$. So by 7.3(1),(2), for some $\zeta < \kappa$ we have: $p_2 \restriction N_\zeta$ is minimal and for some isomorphism f from $N_{\zeta+1}$ onto N_κ we have $f(p_1 \restriction N_\zeta) = p_1$ and $f \restriction N_\zeta = \mathrm{id}_{N_\zeta}$. Also $M_1, N_{\zeta+1}$ are isomorphic over N_ζ (as both are (λ, κ)-saturated over it), hence there is an isomorphism g from $N_{\zeta+1}$ onto M_2 over N_ζ. Now $p_1 = f(p_1 \restriction N_{\zeta+1})$ and $f_2 =: g(p_1 \restriction N_{\zeta+1})$ are non-algebraic extensions of $p_1 \restriction N_\zeta$ which are minimal, hence $p_1 = p_2 \restriction M_1$ and p_2 is as mentioned in 7.5. Now $g \circ f^{-1}$ show that p_1, p_2 are conjugate, so as p_1 is reduced also p_2 is reduced.

(2) Easy, as we can find $N, M_1 \leq_{\mathfrak{K}} N, q \in \mathcal{S}(N)$ extends p_1 and is minimal reduced; without loss of generality $N \leq_{\mathfrak{K}} M_2$ and M_2 is (λ, κ)-saturated over N, and apply part (1). $\blacksquare_{7.5}$

7.6 CLAIM: *Assume*

(a) *$N_{i,j} \in K_\lambda$ for $i \leq \delta_1, j \leq \delta_2$,*
(b) *$\langle N_{i,j} : j \leq \delta_2 \rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous for each $i \leq \delta_1$,*
(c) *$\langle N_{i,j} : i < \delta_1 \rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous for each $j \leq \delta_2$,*
(d) *$\langle N_{i,j} : i \leq \delta_1, j \in \delta_2 \rangle$ is smooth, i.e.*

$$N_{i_1,j_1} \cap N_{i_2,j_2} = N_{\min\{i_1,i_2\}} \cap N_{\min\{j_1,j_2\}},$$

(e) *$N_{i+1,j+1}$ is universal over $N_{i,j+1} \cup N_{i+1,j}$ (i.e. $N_{i+1,j+1}$ is universal over some $N'_{i+1,j+1}$ where $N_{i,j+1} \cup N_{i+1,j} \subseteq N'_{i+1,j+1}$),*
(f) *δ_1 is divisible by $cf(\delta_2) \times \lambda \times \omega$ (and even easier if $\delta_1 = 1!$).*

<u>Then</u> *N_{δ_1,δ_2} is $(\lambda, cf(\delta_1))$-saturated over N_{i,δ_2} for $i < \delta_1$.*

Proof: Without loss of generality $\delta_2 = \mathrm{cf}(\delta_2)$. [Why? Let $\langle \alpha_\varepsilon : \varepsilon \leq \mathrm{cf}(\delta_2) \rangle$ be increasingly continuous with limit δ_2 such that [ε limit $\leftrightarrow \alpha_\varepsilon$ limit], and use $N'_{i,\varepsilon} = N_{i,\alpha_\varepsilon}$.] So δ_1 is divisible by $\delta_2 \times \lambda \times \omega$.

For $i < \delta_1, j < \delta_2$ let $M_{i,j}, M'_{i,j}$ be such that $M_{i+1,j} \cup M_{i,j+1} \subseteq M_{i,j} \leq_{\mathfrak{K}} M'_{i,j} \leq M_{i+1,j+1}$ and $M'_{i,j}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{i,j}$. Clearly $\langle M'_{\varepsilon,\varepsilon} : \varepsilon < \delta_2 \rangle$ is $\leq_{\mathfrak{K}}$-increasing (though not continuous), and $M''_{\varepsilon+1,\varepsilon+1}$ is (λ, κ)-saturated over $M'_{\varepsilon,\varepsilon}$. Let $p^* \in \mathcal{S}(M_{\delta_2,\delta_2})$ be reduced and minimal, so that $M_{\delta_2,\delta_2} = \bigcup_{\varepsilon<\delta_2} M'_{\varepsilon,\varepsilon}$, for some $\varepsilon < \delta_2$, $p^* \restriction M'_{\varepsilon,\varepsilon}$ is minimal hence $p^* \restriction N_{\varepsilon+,\varepsilon+1}$ is minimal, so by renaming $p = p^* \restriction N_{0,0}$ is minimal.

For $i \leq \delta_1, j \leq \delta_2$ let $p_{i,j} \in \mathcal{S}(N_{i,j})$ be the unique non-algebraic extension of p in $\mathcal{S}(N_{i,j})$, so it is minimal. Now for $i < \delta_1$, note that $\langle M'_{i+\varepsilon,\varepsilon} : \varepsilon < \delta_2 \rangle$ is $\leq_{\mathfrak{K}}$-increasing (not continuous!) and $M'_{i+\varepsilon+1,\varepsilon+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M'_{i+\varepsilon,\varepsilon}$ and $\bigcup_{\varepsilon<\delta_2} M'_{i+\varepsilon,\varepsilon} = \bigcup_{\varepsilon<\delta_2} N_{i+\varepsilon+1,\varepsilon+1} = N_{i+\delta_2,\delta_2}$, hence by 7.5(2) we know that $p_{i+\delta_2,\delta_2}$ is reduced (and minimal). In fact, similarly $\alpha < \delta_1$ & $\mathrm{cf}(\alpha) = \mathrm{cf}(\delta_2) \Rightarrow p_{\alpha,\delta_2}$ is reduced. As $N_{i+1,j+1} \neq N_{i+1,j} \cup N_{i,j+1}$ and clause (d) (smoothness) necessarily $N_{i+\delta_2,\delta_2} <_{\mathfrak{K}} N_{i+\delta_2+1,\delta_2}$, hence some $c \in N_{i+\delta_2+1,\delta_2} \setminus N_{i+\delta_2,\delta_2}$ realizes $p_{i+\delta_2,\delta_2}$. So if $\alpha \leq \delta_1$ is divisible by $\delta_2 \times \lambda$ and has cofinality $\mathrm{cf}(\delta_2)$ and $\beta < \alpha$, then by 5.6, N_{α,δ_2} is universal over N_{β,δ_2}. As δ_1 is divisible by $\mathrm{cf}(\delta_2) \times \lambda \times \omega$ we are done. $\blacksquare_{7.6}$

7.7 Lemma: (1) *For every* $N \in K_{\lambda^+}$ *we can find a representation* $\bar{N} = \langle N_i : i < \lambda^+ \rangle$, *with* N_{i+1} *being* $(\lambda, cf(\lambda))$*-saturated over* N_i.

(2) *If for* $\ell = 1, 2$ *we have* $N^\ell = \langle N_i^\ell : i < \lambda^+ \rangle$ *as in part* (1) *<u>then</u> there is an isomorphism* f *from* N^1 *onto* N^2 *mapping* N_i^1 *onto* N_i^2 *for each* $i < \lambda^+$. *Moreover, for any* $i < \lambda^+$ *and isomorphism* g *from* N_i^1 *onto* N_i^2 *we can find an isomorphism* f *from* N^1 *onto* N^2 *extending* g *and mapping* N_j^1 *onto* N_j^2 *for each* $j \in [i, \lambda^+)$.

(3) *If* $N^0 \leq_{\mathfrak{K}} N^1$ *are in* K_{λ^+} *<u>then</u> we can find representations* $\bar{N}^\ell$ *of* N^ℓ *as in* (1) *with* $N_i^0 = N^0 \cap N_i^1$ *(so* $N_i^0 \leq_{\mathfrak{K}} N_i^1$*).*

(4) *For any strictly increasing function* $\mathbf{f} : \lambda^+ \to \lambda^+$, *we can find* $N_{i,\varepsilon}$ *for* $i < \lambda^+, \varepsilon \leq \lambda \times (1 + \mathbf{f}(i))$ *such that:*

(a) $N_{i,\varepsilon} \in K_\lambda$,
(b) $\langle N_{i,\varepsilon} : \varepsilon \leq \lambda \times (1 + \mathbf{f}(i)) \rangle$ *is strictly* $\leq_{\mathfrak{K}}$*-increasing continuous,*
(c) *for each* $\varepsilon, \langle N_{i,\varepsilon} : i \in [i_\varepsilon, \lambda^+) \rangle$ *is a representation as in* (1) *where* $i_\varepsilon = \mathrm{Min}\{i : \varepsilon \leq \lambda \times (1 + \mathbf{f}(i))\}$,
(d) *if* $\varepsilon < \lambda \times (1 + f(i))$ *and* $i < j < \lambda^+$ *<u>then</u>* $N_{j,\varepsilon} \cap N_{i,\lambda\times(1+\mathbf{f}(i))} = N_{i,\varepsilon}$,
(e) $N_{i+1,\varepsilon+1}$ *is* $(\lambda, \aleph_0)$*-saturated over* $N_{i+1,\varepsilon} \cup N_{i,\varepsilon+1}$.

Proof: Straightforward.

(4) First use $\mathbf{f}'$: $\lambda^+ \to \lambda^+$, which is $\mathbf{f}'(i) = \lambda^\omega \times \mathbf{f}(i)$. Then define the $N_{i,\varepsilon}$ for $\varepsilon < \lambda \times (1 + \mathbf{f}'(i))$; $i < \lambda^+$. "Forget" about "$N_{i+1,\varepsilon}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $N_{i,\varepsilon}$"; remember we have disjoint amalgamation by 5.11. Now by 7.6, even for ε

limit divisible by λ^3 we get $N_{i+\lambda,\varepsilon}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $N_{i,\varepsilon}$, so renaming all is O.K. $\blacksquare_{7.7}$

We can deduce the following claim using 7.6, but to keep the door open to other uses we shall not use it.

7.8 CLAIM: *If $\kappa_\ell = \mathrm{cf}(\kappa_\ell) \le \lambda$, and N_ℓ is (λ, κ_ℓ)-saturated over N for $\ell = 1, 2$, then N_1, N_2 are isomorphic over N.*

Proof: We can define by induction on $i \le \lambda \times \kappa_1$, and then by induction on $j \le \lambda \times \kappa_2, M_{i,j}$ such that:

(a) $M_{i,j} \in K_\lambda$,

(b) $M_{0,0} = N$,

(c) $i_1 \le i$ & $j_1 \le j \Rightarrow M_{i_1,j_1} \le_{\mathfrak{K}} M_{i,j}$,

(d) $M_{i_1,j_1} \cap M_{i_2,j_2} = M_{\min\{i_1,i_2\},\min\{j_1,j_2\}}$,

(e) $M_{i,j}$ is $\le_{\mathfrak{K}}$-increasing continuous in i,

(f) $M_{i,j}$ is $\le_{\mathfrak{K}}$-increasing continuous in j,

(g) $M_{0,j} \ne M_{0,j+1}$,

(h) $M_{i+1,0} \ne M_{i,0}$,

(i) $M_{i+1,j+1}$ is universal over $M_{i+1,j} \cup M_{i,j+1}$.

There is no problem with 5.11(1) (using the existence of disjoint amalgamation).

Now $M_{\lambda\times\kappa_1,\lambda\times\kappa_2}$ is the union of the strictly $<_{\mathfrak{K}}$-increasing sequence $\langle M_{0,0}\rangle\hat{}\langle M_{\lambda\times i,\lambda\times\kappa_2} : i < \kappa_1\rangle$, hence by 7.6 is (λ, κ_1)-saturated over $M_{0,0} = N$, hence $M_{\lambda\times\kappa_1,\lambda\times\kappa_2} \cong_N N_1$. Similarly $M_{\lambda\times\kappa_1,\lambda\times\kappa_2}$ is the union of the strictly $\le_{\mathfrak{K}}$-increasing sequence $\langle M_{0,0}\rangle\hat{}\langle M_{\lambda\times\kappa_1,\lambda\times j} : j < \kappa_2\rangle$, hence is (λ, κ_2)-saturated over $M_{0,0} = N$, hence $M_{\lambda\times\kappa_1,\lambda\times\kappa_2} \cong_N N_2$. Together N_1, N_2 are isomorphic over N. $\blacksquare_{7.8}$

7.9 CLAIM: *For any $M^* <_{\mathfrak{K}} N^*$ in K_λ we can find υ, an ordinal power of λ which is $< \lambda^+$ and $\langle M_1 : i \le \upsilon\rangle$, $\le_{\mathfrak{K}}$-increasing continuous st $(M_i, M_{i+1}) \cong (M^*, N^*)$ and M_υ is (λ, υ)-saturated over M_i for every $i < \upsilon$.*

Proof: By the categoricity in λ.

8. Uniqueness of amalgamation in $\mathfrak{K}_\lambda$

We deal in this section only with K_λ. We want to, at least, approximate unique amalgamation using as starting point $\otimes$ of 6.4 (see also 6.7), i.e. $K_\lambda^{3,\mathrm{uq}} \ne \emptyset$.

8.1 HYPOTHESIS: (1) Assume hypothesis 7.1, so

(a) $\mathfrak{K}$ is an abstract elementary class,

(b) $\mathfrak{K}$ has amalgamation in λ,

(c) $\mathfrak{K}$ is categorical in λ (can be weakened),
(d) $\mathfrak{K}$ is stable in λ (see 5.7, clause (a)),
(e) there is an inevitable $p \in \mathcal{S}(N)$ for $N \in K_\lambda$ (holds by 5.3),
(f) the basic properties in type theory.

(2) (M^*, N^*) is some pair in $K_\lambda^{3,\mathrm{uq}} = \{(M_0, M_2) : M_0 <_{\mathfrak{K}} M_2$ are in K_λ and for every $M_1, M_0 \le_{\mathfrak{K}} M_1 \in K_\lambda \Rightarrow (M_0, M_1, M_2) \in K_\lambda^{\mathrm{uniq}}$; equivalently for some $M_1, (M_0, M_1, M_2)$ are as in $\bigotimes$ of 6.4$\}$ (eventually the choice does not matter if each time, instead of $\cong (M^*, N^*)$, we write $\in K_\lambda^{2,\mathrm{uq}}$, see 8.11; but if we start with this definition then the uniqueness theorems will be more cumbersome).

(3) Lastly let υ be as in 7.9 for our (M^*, N^*).

8.2 Definition: Assume $\bar\delta = \langle \delta_1, \delta_2, \delta_3 \rangle, \delta_1$ a limit ordinal $< \lambda^+$ but δ_2, δ_3 are $< \lambda^+$ and may be 0. We say that $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ (we say N_1, N_2 are **saturated by and smoothly amalgamated** in N_3 over N_0 for $\bar\delta$) when:

(a) $N_\ell \in K_\lambda$ for $\ell \in \{0,1,2,3\}$,
(b) $N_0 \le_{\mathfrak{K}} N_\ell \le_{\mathfrak{K}} N_3$ for $\ell = 1, 2$,
(c) $N_1 \cap N_2 = N_0$ (i.e. in disjoint amalgamation),
(d) N_1 is (λ,cf(δ_1))-saturated over N_0,
(e) N_2 is (λ,cf(δ_2))-saturated over N_0; if $\delta_2 = 1$ this means just $N_0 \le_{\mathfrak{K}} N_2$,
(f) there are $N_{1,i}, N_{2,i}$ for $i \le \upsilon \times \delta_1$ (called the witness) such that:
  (α) $N_{1,0} = N_0, N_{1,\lambda\times\delta_1} = N_1$,
  (β) $N_{2,0} = N_2$,
  (γ) $\langle N_{\ell,i} : i \le \upsilon \times \delta_1 \rangle$ is $<_{\mathfrak{K}}$-increasing continuous for $\ell = 1, 2$,
  (δ) $(N_{1,i}, N_{1,i+1}) \cong (M^*, N^*)$,
  (ε) $N_{2,i} \cap N_1 = N_{1,i}$,
  (ζ) N_3 is (λ,cf(δ_3))-saturated over $N_{2,\upsilon\times\delta_1}$; if $\delta_3 = 1$ this means just $N_{2,\upsilon\times\delta_1} \le_{\mathfrak{K}} N_3$.

Discussion: Why this definition of NF? We need a nonforking notion with the usual properties. We first describe a version depending on $\langle \delta_0, \delta_1, \delta_2 \rangle$ and get $NF = NF_{\lambda,\bar\delta}$; $\bar\delta$ works like a scaffold—eventually $\bar\delta$ disappears. Clearly if there is such a notion, it should agree with the Definitions 8.2 and 8.3.

8.3 Definition: (1) We say $N_1 \bigcup_{N_0}^{N_3} N_2$ (or N_1, N_2 are **smoothly amalgamated** over N_0 inside N_3 or $NF_\lambda(N_0, N_1, N_2, N_3)$) <u>if</u> we can find $M_\ell \in K_\lambda$ (for $\ell < 4$) such that:

(a) $NF_{\lambda,\langle\lambda,\lambda,\lambda\rangle}(M_0, M_1, M_2, M_3)$,

(b) $N_\ell \leq_{\mathfrak{K}} M_\ell$ for $\ell < 4$,
(c) $N_0 = M_0$,
(d) M_1, M_2 are $(\lambda, \mathrm{cf}(\lambda))$-saturated over N_0 (follows by (a); see clauses (d), (e) of 8.2).

8.4 Claim: (1) *Assume $\bar{\delta} = \langle \delta_1, \delta_2, \delta_3 \rangle, \delta_\ell$ a limit ordinal $< \lambda^+$ and $N_\ell \in K_\lambda$ for $\ell < 3$, and N_1 is $(\lambda, cf(\delta_1))$-saturated over N_0 and N_2 is $(\lambda, cf(\delta_2))$-saturated over N_0 and $N_0 \leq_{\mathfrak{K}} N_1, N_0 \leq_{\mathfrak{K}} N_2$ and for simplicity $N_1 \cap N_2 = N_0$. Then we can find N_3 such that $NF_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$.*
(2) *Moreover, we can choose any $\langle N_{1,i} : i \leq \upsilon \times \delta_1 \rangle$ as in 8.2(f)$(\alpha),(\gamma),(\delta)$ as part of the witness.*

Proof: Straightforward (remembering 5.6, 7.9 (and uniqueness of the $(\lambda, \mathrm{cf}(\delta_1))$-saturated model over N_0)). $\blacksquare_{8.4}$

8.5 Claim: *In Definition 8.2, if δ_3 is a limit ordinal, then without loss of generality (even without changing $\langle N_{1,i} : i \leq \upsilon \times \delta_1 \rangle$)*
(g) *$N_{2,i+1}$ is $(\lambda, cf(\delta_2))$-saturated over $N^1_{i+1} \cup N^2_i$ (which means it is $(\lambda, cf(\delta_2))$-saturated over some N, where $N^1_{i+1} \cup N^2_i \subseteq N \leq_{\mathfrak{K}} N_{2,i+1}$).*

Proof: So assume $NF_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$ holds as witnessed by $\langle N_{\ell,i} : i \leq \upsilon \times \delta_\ell \rangle$ for $\ell = 1, 2$. Now we choose by induction on $i \leq \upsilon \times \delta_1$ a model $M_{2,i} \in K_\lambda$ such that:
(i) $N_{2,i} \leq M_{2,i}$,
(ii) $M_{2,0} = N_2$,
(iii) $M_{2,i}$ is $\leq_{\mathfrak{K}}$-increasing continuous,
(iv) $M_{2,i} \cap N_{2,\upsilon \times \delta_1} = N_{2,i}$, moreover $M_{2,i} \cap N_3 = N_{2,i}$,
(v) $M_{2,i+1}$ is $(\lambda, \mathrm{cf}(\delta_2))$-saturated over $M_{2,i} \cup N_{2,i+1}$.

There is no problem to carry the definition. Let M_3 be such that $M_{2,\upsilon \times \delta_1} \leq_{\mathfrak{K}} M_3 \in K_\lambda$ and M_3 is $(\lambda, \mathrm{cf}(\delta_3))$-saturated over $M_{2,\lambda \times \delta_1}$. So both M_3 and N_3 are $(\lambda, \mathrm{cf}(\delta_3))$-saturated over $N_{2,\upsilon \times \delta_1}$, hence they are isomorphic over $N_{2,\upsilon \times \delta_1}$, so let f be an isomorphism from M_3 onto N_3 which is the identity over $N_{2,\upsilon \times \delta_1}$. Clearly $\langle N_{1,i} : i \leq \upsilon \times \delta_1 \rangle, \langle f(M_{2,i}) : i \leq \upsilon \times \delta_1 \rangle$ are also witnesses for $NF_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$ satisfying the extra demand (g). $\blacksquare_{8.5}$

8.6 Claim (Weak Uniqueness): *Assume that for $x \in \{a, b\}$, we have $NF_{\lambda,\bar{\delta}^x}(N^x_0, N^x_1, N^x_2, N^x_3)$ as witnessed by $\langle N^x_{1,i} : i \leq \upsilon \times \delta^x_1 \rangle$, $\langle N^x_{2,i} : i \leq \upsilon \times \delta^x_1 \rangle$ and $\delta_1 =: \delta^a_1 = \delta^b_1$, $cf(\delta^a_2) = cf(\delta^b_2)$ and $cf(\delta^a_3) = cf(\delta^b_3) \geq \aleph_0$.*

Suppose further that f_ℓ is an isomorphism from N^a_ℓ onto N^b_ℓ for $\ell = 0, 1, 2$; moreover: $f_0 \subseteq f_1, f_0 \subseteq f_2$ and $f_1(N^a_{1,i}) = N^b_{1,i}$.

Then we can find an isomorphism f from N_3^a onto N_3^b extending $f_1 \cup f_2$.

Proof: Without loss of generality $N^x_{2,i+1}$ is $(\lambda, \mathrm{cf}(\delta_2^*))$-saturated over $N^x_{1,i+1} \cup N^x_{2,i}$ (by 8.5, note the "without changing the $N_{1,i}$'s" there). Now we choose by induction on $i \leq \upsilon \times \delta_1$ an isomorphism g_i from $N^a_{2,i}$ onto $N^b_{2,i}$ such that: g_i is increasing in i and g_i extends $(f_1 \restriction N^a_{1,i}) \cup f_2$.

For $i = 0$ choose $g_0 = f_2$ and for i limit let g_i be $\bigcup_{j<i} g_j$ and for $i = j+1$ use $(N_{1,i}, N_{1,i+1}) \cong (M^*, N^*)$ (see 8.2) and the extra saturation clause (g). Now we can extend $g_{\lambda\times\delta_1}$ to an isomorphism from N_3^a onto N_3^b as N_3^x is $(\lambda, \mathrm{cf}(\delta_3))$-saturated from $N^x_{2,\upsilon\times\delta_1}$ (for $x \in \{a,b\}$); note that, knowing 8.6, possibly the choice of $\langle N_{1,i} : i \leq \upsilon \times \delta_1 \rangle$ matters. $\blacksquare_{8.6}$

Now we prove an "inverted" uniqueness

8.7 CLAIM: *Suppose that*

(a) *for $x \in \{a,b\}$ we have $NF_{\lambda,\bar\delta^x}(N_0^x, N_1^x, N_2^x, N_3^x)$,*
(b) *$\bar\delta^x = \langle \delta_1^x, \delta_2^x, \delta_3^x \rangle$, $\delta_1^a = \delta_2^b$, $\delta_2^a = \delta_1^b$, $cf(\delta_3^a) = cf(\delta_3^b)$, all limit ordinals,*
(c) *f_0 is an isomorphism from N_0^a onto N_0^b,*
(d) *f_1 is an isomorphism from N_1^a onto N_2^b,*
(e) *f_2 is an isomorphism from N_2^a onto N_1^b,*
(f) *$f_0 \subseteq f_1$ and $f_0 \subseteq f_2$.*

Then there is an isomorphism from N_3^a onto N_3^b extending $f_1 \cup f_2$.

Before proving this claim, we have

8.8 SUBCLAIM: (1) *For any limit ordinals $\delta_1^a, \delta_2^a, \delta_3^a \leq \lambda$ we can find $M_{i,j}$ (for $i \leq \upsilon \times \delta_1^a$ and $j \leq \upsilon \times \delta_2^a$) and M_3 such that:*

(A) *$M_{i,j} \in K_\lambda$,*
(B) *$i_1 \leq i_2$ & $j_1 \leq j_2 \Rightarrow M_{i_1,j_1} \leq_{\mathfrak{K}} M_{i_2,j_2}$,*
(C) *if $i \leq \upsilon \times \delta_1$ is a limit ordinal and $j \leq \upsilon \times \delta_2$ then $M_{i,j} = \bigcup_{\zeta<i} M_{\zeta,j}$,*
(D) *if $i \leq \upsilon \times \delta_1$ and $j \leq \upsilon \times \delta_2$ is a limit ordinal then $M_{i,j} = \bigcup_{\xi<j} M_{i,\xi}$,*
(E) *$(M_{0,j}, M_{0,j+1}) \cong (M^*, N^*)$,*
(F) *$(M_{i,0}, M_{i+1,0}) \cong (M^*, N^*)$,*
(G) *for $i_1, i_2 \leq \upsilon \times \delta_1$ and $j_1, j_2 \leq \upsilon \times \delta_2$ we have*

$$M_{i_1,j_1} \cap M_{i_2,j_2} = M_{\min\{i_1,i_2\},\min\{j_1,j_2\}},$$

(H) *$M_{\upsilon\times\delta_1,\upsilon\times\delta_2} \leq_{\mathfrak{K}} M_3 \in K_\lambda$ moreover M_3 is $(\lambda, cf(\delta_3))$-saturated over $M_{\upsilon\times\delta_1,\upsilon\times\delta_2}$,*
(I) *$M_{0,j+\upsilon}$ is (λ, υ')-saturated over $M_{0,j}$ and $M_{i+\upsilon,0}$ is (λ,υ)-saturated over $M_{i,0}$ for $i < \upsilon \times \delta_1$, $j < \upsilon \times \delta_2$.*

(2) *Moreover, it is O.K. if* $\langle M_{0,j} : j \le \upsilon \times \delta_2^a\rangle, \langle M_{i,0} : i \le \upsilon \times \delta_1^a\rangle$ *are pregiven as long as both are* $\le_{\mathfrak{K}}$*-increasing continuous in* $\mathfrak{K}_\lambda$ *satisfying* (E)+(F)+(I) *and* $M_{0,\upsilon\times\delta_2^a} \cap M_{\upsilon\times\delta_2^a,0} = M_{0,0}$.

Proof: (1) For $i = 0$ and for $j = 0$ this is done by 7.9. Otherwise this is done by induction on i and for fixed i by induction on j. For i limit use clause (C) (and check). For j limit use clause (D) and if $j = \xi + 1$ use the existence of disjoint amalgamation (i.e. 5.11).

Lastly, choose $M_3 \in K_\lambda$ which is $(\lambda, \mathrm{cf}(\delta_3^a))$-saturated over $M_{\upsilon\times\delta_1^a,\upsilon\times\delta_2^a}$.

(2) Similar to (1). $\blacksquare_{8.8}$

Proof of 8.7: Let $M_{i,j}, M_3$ be as in 8.8. For $x \in \{a,b\}$ as $NF_{\lambda,\bar\delta^x}(N_0^x, N_1^x, N_2^x, N_3^x)$, we know that there are witnesses $\langle N_{1,i}^x : i \le \upsilon \times \delta_1^x\rangle$, $\langle N_{2,i}^x : i \le \upsilon \times \delta_1^x\rangle$ for this, so $\langle N_{1,i}^x : i \le \upsilon \times \delta_1^x\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous and $(N_{1,i}^x, N_{1,i+1}^x) \cong (M^*, N^*)$. Therefore $\langle N_{1,i}^a : i \le \upsilon\times\delta_1^a\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous sequences with each successive pair isomorphic to (M^*, N^*), hence by 8.8(2), without loss of generality, there is an isomorphism g_1 from $N_{1,\upsilon\times\delta_1^a}^a$ onto $M_{\upsilon\times\delta_1^a}$, mapping $N_{1,i}^a$ onto $M_{i,0}$; remember $N_{1,\upsilon\times\delta_1^a}^a = N_1^a$. Let $g_0 = g_1 \restriction N_0^a = g_1 \restriction N_{1,0}^a$, so $g_0 \circ f_0^{-1}$ is an isomorphism from N_0^b onto $M_{0,0}$.

As $\delta_1^b = \delta_2^a$, using 8.8(2) fully, without loss of generality there is an isomorphism g_2 from $N_{1,\upsilon\times\delta_2^a}^b$ onto $M_{0,\upsilon\times\delta_2^a}$ mapping $N_{1,j}^b$ onto $M_{0,j}$ (for $j \le \upsilon \times \delta_2^a$) and g_2 extends $g_0 \circ f_0^{-1}$.

Now we want to use the weak uniqueness 8.6 and for this note:

(α) $NF_{\lambda,\bar\delta^a}(N_0^a, N_1^a, N_2^a, N_3^a)$ as witnessed by $\langle N_{1,i}^a : i \le \upsilon \times \delta_1^a\rangle$, $\langle N_{2,i}^a : i \le \upsilon \times \delta_1\rangle$.
[Why? An assumption.]

(β) $NF_{\lambda,\bar\delta^a}(M_{0,0}, M_{\upsilon\times\delta_1^a,0}, M_{0,\upsilon\times\delta_2^a}, M_3)$ as witnessed by the sequences $\langle M_{i,0} : i \le \upsilon \times \delta_1\rangle, \langle M_{i,\upsilon\times\delta_2^a} : i \le \upsilon \times \delta_2^a\rangle$.
[Why? Check.]

(γ) g_0 is an isomorphism from N_0^a onto $M_{0,0}$.
[Why? See its choice.]

(δ) g_1 is an isomorphism from N_1^a onto $M_{\upsilon\times\delta_1^a,0}$ mapping $N_{1,i}^a$ onto $M_{i,0}$ for $i \le \upsilon \times \delta_1^a$ and extending g_0.
[Why? See the choice of g_1 and of g_0.]

(ε) $g_2 \circ f_2$ is an isomorphism from N_2^a onto $M_{0,\upsilon\times\delta_2^a}$ extending g_0.
[Why? f_2 is an isomorphism from N_2^a onto N_1^b and g_2 is an isomorphism from N_1^b onto $M_{0,\upsilon\times\delta_1^a}$ extending $g_0 \circ f_0^{-1}$ and $f_0 \subseteq f_2$.]

So by 8.6 there is an isomorphism g_3^a from N_3^a onto M_3 extending g_1 and $g_2 \circ f_2$.

We next want to apply 8.6 to the N_l^b's; so note:

$(\alpha)'$ $NF_{\lambda,\bar{\delta}^b}(N_0^b, N_1^b, N_2^b, N_3^b)$ as witnessed by $\langle N_{1,i}^b : i \leq \upsilon \times \delta_2^a\rangle$, $\langle N_{2,i}^b : i \leq \upsilon \times \delta_2^a\rangle$.

$(\beta)'$ $NF_{\lambda,\bar{\delta}^b}(M_{0,0}, M_{0,\upsilon\times\delta_2^a}, M_{\upsilon\times\delta_1^a,0}, M_3)$ as witnessed by the sequences $\langle M_{0,j} : j \leq \upsilon \times \delta_2^a\rangle$, $\langle M_{\upsilon\times\delta_1^a,j} : j \leq \upsilon \times \delta_1^a\rangle$.

$(\gamma)'$ $g_0 \circ (f_0)^{-1}$ is an isomorphism from N_0^b onto $M_{0,0}$.
[Why? Check.]

$(\delta)'$ g_2 is an isomorphism from N_1^b onto $M_{0,\upsilon\times\delta_2^a}$ mapping $N_{1,j}^b$ onto $M_{0,j}$ for $j \leq \upsilon \times \delta_2^a$ and extending $g_0 \circ (f_1)^{-1}$.
[Why? See the choice of g_2: it maps $N_{1,j}^b$ onto $M_{0,j}$.]

$(\varepsilon)'$ $g_1 \circ (f_1)^{-1}$ is an isomorphism from N_2^b onto $M_{\upsilon\times\delta_0^a}$ extending g_0.
[Why? Remember f_1 is an isomorphism from N_1^a onto N_2^b extending f_0 and the choice of g_1: it maps N_1^a onto $M_{\upsilon\times\delta_{1,0}^a}$.]

So there is an isomorphism g_3^b from N_3^b onto M_3 extending $g_2, f_1 \circ (f_1)^{-1}$.

Lastly, $(g_3^b)^{-1} \circ g_3^a$ is an isomorphism from N_3^a onto N_3^b (chase arrows). Also

$$\begin{aligned}((g_b^b)^{-1} \circ g_3^a) \restriction N_1^a &= (g_3^b)^{-1}(g_3^a \restriction N_1^a)\\ &= (g_3^b)^{-1} g_1 = ((g_3^b)^{-1} \restriction M_{\upsilon\times\delta_{1,0}^a}) \circ g_1\\ &= (g_3^b \restriction N_2^b)^{-1} \circ g_1 = ((g_1 \circ (f_1)^{-1})^{-1}) \circ g_1\\ &= (f_1 \circ (g_1)^{-1}) \circ g_1 = f_0.\end{aligned}$$

Similarly $((g_3^b)^{-1} \circ g_3^a) \restriction N_2^a = f_2$. So we have finished. $\blacksquare_{8.7}$

8.9 Claim (Uniqueness): *Assume for $x \in \{a,b\}$ we have $NF_{\lambda,\bar{\delta}^x}(N_0^x, N_1^x, N_2^x, N_3^x)$ and $cf(\delta_1^a) = cf(\delta_1^b), cf(\delta_2^a) = cf(\delta_2^b), cf(\delta_3^a) = cf(\delta_3^b)$, all δ_ℓ^x limit ordinals.*

If f_ℓ is an isomorphism from N_ℓ^a onto N_ℓ^b for $\ell < 3$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ __then__ there is an isomorphism f from N_3^a onto N_3^b extending f_1, f_2.

Proof: Let $\bar{\delta}^c = \langle \delta_1^c, \delta_2^c, \delta_3^c\rangle = \langle \delta_2^a, \delta_1^a, \delta_3^a\rangle$; by 8.4 there are N_ℓ^c (for $\ell \leq 3$) such that $NF_{\lambda,\bar{\delta}^c}(N_0^c, N_1^c, N_2^c, N_3^c)$. There is for $x \in \{a,b\}$ an isomorphism g_0^x from N_0^a onto N_0^c (as K_λ is categorical in λ) and, without loss of generality, $g_0^b = g_0^a \circ f_0$. Similarly, for $x \in \{a,b\}$ there is an isomorphism g_1^x from N_1^x onto N_2^c extending g_0^x (as N_1^x is $(\lambda, \mathrm{cf}(\delta_1^x))$-saturated over N_0^x and also N_2^c is $(\lambda, \mathrm{cf}(\delta_2^c))$-saturated over N_0^c and $\mathrm{cf}(\delta_2^c) = \mathrm{cf}(\delta_1^a) = \mathrm{cf}(\delta_1^x)$) and, without loss of generality, $g_1^b = g_1^a \circ f_1$. Similarly, for $x \in \{a,b\}$ there is an isomorphism g_2^x from N_2^x onto N_1^c extending g_0^x (as N_2^x is $(\lambda, \mathrm{cf}(\delta_2^x))$-saturated over N_0^x and also N_1^c is $(\lambda, \mathrm{cf}(\delta_1^c))$-saturated over N_0^c and $\mathrm{cf}(\delta_1^c) = \mathrm{cf}(\delta_2^a) = \mathrm{cf}(\delta_2^x)$) and, without loss of generality, $g_2^b = g_2^b \circ f_2$.

So by 8.7 for $x \in \{a,b\}$ there is an isomorphism g_3^x from N_3^x onto N_3^c extending g_1^x and g_2^x. Now $(g_3^b)^{-1} \circ g_3^a$ is an isomorphism from N_3^a onto N_3^b extending f_1, f_2 as required. $\blacksquare_{8.9}$

8.10 CONCLUSION (Symmetry): If $NF_{\lambda,\langle\delta_1,\delta_2,\delta_3\rangle}(N_0,N_1,N_2,N_3)$ <u>then</u> $NF_{\lambda,\langle\delta_2,\delta_1,\delta_3\rangle}(N_0,N_2,N_1,N_3)$.

Proof: By 8.7 (and 8.9).

8.11 CLAIM: *In Definition* 8.2 *we can replace* $(N_{1,i},N_{1,i+1}) \cong (M^*,N^*)$ *by* $(N_{1,i},N_{1,i+1}) \in K^{3,\mathrm{uq}}$.

Proof: Like the proof of 8.7 (get $(M_{i,0},M_{i+1,0}) \cong (M^*,N^*)$, $(M_{0,j},M_{0,j+1}) \in K^{3,\mathrm{uq}}_\lambda$), but as we shall not use it, we do not elaborate. $\blacksquare_{8.11}$

Now we turn to smooth amalgamation (not necessarily saturated, see Definition 8.3).

8.12 CLAIM: (1) *If* $NF_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ *and each* δ_ℓ *is limit* <u>then</u> $NF_\lambda(N_0,N_1,N_2,N_3)$ *(see Definition* 8.3*).*
(2) *In Definition* 8.3 *we can add:*
(d)$^+$ M_ℓ *is* $(\lambda,\mathrm{cf}(\lambda))$*-saturated over* N_0 *and, moreover, over* N_ℓ,
(e) M_3 *is* $(\lambda,\mathrm{cf}(\lambda))$*-saturated over* $M_1\cup M_2$ *(actually, this is given by* $(f)(\zeta)$ *of Definition* 8.2*).*

Proof: (1), (2). By 8.8 we can find $M_{i,j}$ for $i \le \upsilon\times(\delta_1+\lambda)$, $j \le \upsilon\times(\delta_2+\lambda)$ for $\bar\delta' =: \langle\delta_1+\lambda,\delta_2+\lambda,\delta_3\rangle$ and choose $M_3' \in K_\lambda$ which is $(\lambda,\mathrm{cf}(\delta_3))$-saturated over $M_{\upsilon\times\delta_1,\upsilon\times\delta_2}$. So $NF_{\lambda,\bar\delta}(M_{0,0},M_{\upsilon\times\delta_1,0},M_{0,\upsilon\times\delta_2},M_3')$; hence by 8.9, without loss of generality, $M_{0,0}=N_0, M_{\upsilon\times\delta_1,0}=N_1, M_{0,\upsilon\times\delta_2}=N_2$, and $N_3 = M_3'$. Lastly, let M_3 be $(\lambda,\mathrm{cf}(\lambda))$-saturated over M_3'. Now clearly also $NF_{\lambda,\langle\delta_1+\lambda,\delta_2+\lambda,\delta_3+\lambda\rangle}(M_{0,0},M_{\upsilon\times(\delta_1+\lambda),0},M_{0,\upsilon\times(\delta_2+\lambda)},M_3)$ and $N_0 = M_{0,0}, N_1 = M_{\upsilon\times\delta_2,0} \le_{\mathfrak{K}} M_{\upsilon\times(\delta_2+\lambda),0}, N_2 = M_{0,\upsilon\times\delta_2} \le_{\mathfrak{K}} M_{0,\upsilon\times(\delta_2+\lambda)}$ and $M_{\upsilon\times(\delta_1+\lambda),0}$ is $(\lambda,\mathrm{cf}(\lambda))$-saturated over $M_{\upsilon\times\delta_1,0}$ and $M_{0,\upsilon\times(\delta_2+\lambda)}$ is $(\lambda,\mathrm{cf}(\lambda))$-saturated over $M_{0,\upsilon\times\delta_2}$ and $N_3 = M_3' \le_{\mathfrak{K}} M_3$. So we get all the requirements for $NF_\lambda(N_0,N_1,N_2,N_3)$ (as witnessed by $\langle M_{0,0},M_{\upsilon\times(\delta_1+\lambda),0},M_{0,\upsilon\times(\delta_2+\lambda)},M_3\rangle$). $\blacksquare_{8.12}$

8.13 CLAIM (Uniqueness of smooth amalgamation): *If* $NF_\lambda(N_0^x,N_1^x,N_2^x,N_3^x)$ *for* $x\in\{a,b\}$, f_ℓ *an isomorphism from* N_ℓ^a *onto* N_ℓ^b *for* $\ell<3$ *and* $f_0\subseteq f_1, f_0\subseteq f_2$ <u>*then*</u> $f_1\cup f_2$ *can be extended to a* $\le_{\mathfrak{K}}$*-embedding of* N_3^a *into some* $\le_{\mathfrak{K}}$*-extension of* N_3^b, *so if* N_3^x *is* (λ,κ)*-saturated over* $N_1^x\cup N_2^x$ *for* $x=a,b$, *we can extend* $f_1\cup f_2$ *to an isomorphism from* N_3^a *onto* N_3^b.

Proof: For $x\in\{a,b\}$ let the sequence $\langle M_\ell^x : \ell<4\rangle$ be a witness to $NF_\lambda(N_0^x,N_1^x,N_2^x,N_3^x)$ as in 8.3, 8.12(2), so in particular

$NF_{\lambda,\langle\lambda,\lambda,\lambda\rangle}(M_0^x, M_1^x, M_2^x, M_3^x)$. By chasing arrows and uniqueness, i.e. 8.7, without loss of generality $M_\ell^a = M_\ell^b$ for $\ell < 4$ and $f_0 = \mathrm{id}_{N_0^a}$. As M_1^a is $(\lambda, \mathrm{cf}(\lambda))$-saturated over N_1^a and also over N_1^b and f_1 is an isomorphism from N_1^a onto N_1^b, clearly there is an automorphism g_1 of M_1^a such that $f_1 \subseteq g_1$, hence also $\mathrm{id}_{N_0^a} = f_0 \subseteq f_1 \subseteq g_1$. Similarly there is an automorphism g_2 of M_2^a extending f_2 hence f_0. So $g_\ell \in \mathrm{AUT}(M_\ell^a)$ for $\ell = 1, 2$ and $g_1 \restriction M_0^a = f_0 = g_2 \restriction M_0^a$. By the uniqueness of $NF_{\lambda,\langle\lambda,\lambda,\lambda\rangle}$ (i.e. Claim 8.9) there is an automorphism g_4 of M_4^a extending $g_1 \cup g_2$. This proves the desired conclusion. $\blacksquare_{8.13}$

8.14 CLAIM: *Assume*

(a) $\bar\delta = \langle\delta_1, \delta_2, \delta_3\rangle, \delta_\ell < \lambda^+$ *is a limit ordinal for* $\ell = 1, 2, 3$; $N_0 \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} N_3$ *for* $\ell = 1, 2$ *and*

(b) N_ℓ *is* $(\lambda, cf(\delta_\ell))$*-saturated over* N_0 *for* $\ell = 1, 2$,

(c) N_3 *is* $cf(\delta_3)$*-saturated over* $N_1 \cup N_2$.

<u>*Then*</u> $NF_\lambda(N_0, N_1, N_2, N_3)$ *iff* $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$.

Proof: The "if" direction holds by 8.12(1). For the "only if" direction, by the proof of 8.12(1) (and Definitions 8.2, 8.3) we can find M_ℓ ($\ell \leq 3$) such that $NF_{\lambda,\bar\delta}(M_0, M_1, M_2, M_3)$ and clauses (b), (c), (d) of Definition 8.3 hold, so by 8.12 also $NF_\lambda(M_0, M_1, M_2, M_3)$. Easily, there is for $\ell < 3$ an isomorphism f_ℓ from M_ℓ onto N_ℓ such that $f_0 = f_\ell \restriction M_\ell$. By the uniqueness for smooth amalgamation (i.e. 8.13) we can find an isomorphism f_3 from M_3 onto N_3 extending $f_1 \cup f_2$. So as $NF_{\lambda,\bar\delta}(M_0, M_1, M_2, M_3)$, also $NF_{\lambda,\bar\delta},(f_0(M_0), f_3(M_1), f_3(M_2), f_3(M_3))$; i.e. $NF_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ as required. $\blacksquare_{8.14}$

8.15 CLAIM (Monotonicity): *If* $NF_\lambda(N_0, N_1, N_2, N_3)$ *and* $N_0 \leq_{\mathfrak{K}} N_1' \leq_{\mathfrak{K}} N_1$ *and* $N_0 \leq_{\mathfrak{K}} N_2' \leq_{\mathfrak{K}} N_2$ *and* $N_1' \cup N_2' \subseteq N_3' \leq_{\mathfrak{K}} N_3$ <u>*then*</u> $NF_\lambda(N_0, N_1', N_2', N_3')$.

Proof: Read Definition 8.3.

8.16 CLAIM (Symmetry): $NF_\lambda(N_0, N_1, N_2, N_3)$ *holds* <u>*if and only if*</u> $NF_\lambda(N_0, N_2, N_1, N_3)$ *holds.*

Proof: By Claim 8.10 (and Definition 8.3).

8.17 CLAIM: *Assume* $\alpha < \lambda^+$ *is an ordinal and for* $x \in \{a, b, c\}$ *the sequence* $\langle N_i^x : i \leq \alpha\rangle$ *is a* $\leq_{\mathfrak{K}}$*-increasing sequence of members of* K_λ, *for* $x = a, b$ *the sequence is* $\leq_{\mathfrak{K}}$*-increasing continuous for* $i \leq \alpha, N_i^b \cap N_\alpha^a = N_i^a$, $N_i^c \cap N_\alpha^a = N_i^a, N_i^a \leq_{\mathfrak{K}} N_i^b \leq_{\mathfrak{K}} N_i^c$ *and* N_i^c *is* (λ, κ_i)*-saturated over* N_i^b *and* $NF_{\lambda,\bar\delta^i}(N_i^a, N_{i+1}^a, N_i^c, N_{i+1}^b)$ *(so* $i < \alpha \Rightarrow N_i^c \leq_{\mathfrak{K}} N_{i+1}^b$*) where* $\bar\delta^i = \langle\delta_1^i, \delta_2^i, \delta_3^i\rangle$ *sequence of limit ordinals,* $i < \alpha \Rightarrow \delta_2^{i+1} = \delta_i^3$, *and for* $i < 0$ *limit,* $cf(\delta_i^3) =$

$\sum_{j<i}\delta^3_j, \delta_1 = \sum_{\beta<\alpha}\delta^1_\beta$ *and* $\delta_3 = \kappa_\alpha, \bar{\delta} = \langle \delta_1, \delta^0_2, \delta_3\rangle$. *Then* $NF_{\lambda,\bar{\delta}}(N^a_0, N^a_\alpha, N^b_0, N^c_\alpha)$.

Proof: Use uniqueness of 8.9; lastly use 8.9 to show N^b_α is $(\lambda, \mathrm{cf}(\alpha))$-saturated over $N^a_\alpha \cup N^b_0$. $\blacksquare_{8.17}$

8.18 CLAIM: *Assume that $\alpha < \lambda^+$ and for $x \in \{a,b\}$ we have $\langle N^x_i : i \leq \alpha\rangle$ is a $\leq_{\mathfrak{K}}$-increasing continuous sequence of members of K_λ.*

(1) *If $NF_\lambda(N^a_i, N^a_{i+1}, N^b_i, N^b_{i+1})$ for each $i < \alpha$ then $NF_\lambda(N^a_0, N^a_\alpha, N^b_0, N^b_\alpha)$.*

(2) *If $\alpha_1 < \lambda^+, \alpha_2 < \lambda^+$ and $M_{i,j}$ $(i \leq \alpha_1, j \leq \alpha_2)$ satisfy clauses (A), (B), (C), (D) of 8.8, and for each $i < \alpha_1, j < \alpha_2$ we have:*

$$M_{i,j+1} \quad \underset{M_{i,j}}{\overset{M_{i+1,j+1}}{\bigcup\!\!\!\!\bigcup}} \quad M_{i+1,j},$$

then

$$M_{i,0} \quad \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup\!\!\!\!\bigcup}} \quad M_{0,j} \quad \text{for } i \leq \alpha_1, j \leq \alpha_1.$$

Proof: (1) We first prove special cases and use them to prove more general cases. Case A: N^a_{i+1} is (λ, δ^1_i)-saturated over N^a_i and N^b_{i+1} is (λ, δ^2_i)-saturated over $N^a_{i+1} \cup N^b_i$ for $i < \alpha$.

We can choose, for $i \leq \alpha, N^c_i \in K_\lambda$ such that

(a) $N^b_i \leq_{\mathfrak{K}} N^c_i \leq_{\mathfrak{K}} N^b_{i+1}, N^c_i$ is (λ, δ^2_0)-saturated over N^b_i, and $NF_{\lambda,\langle\delta^0_1,\delta^0_2,\delta^0_2\rangle}(N^a_i, N^a_{i+1}, N^c_i, N^b_{i+1})$,

(b) $N^c_\alpha \in K_\lambda$ is $(\lambda, \delta^\alpha_3)$-saturated over N^b_α.

(Possible by uniqueness, i.e. 8.13, and monotonicity, i.e. 8.15). Now we can use 8.17.

Case B: For each $i < \alpha$ we have: N^a_{i+1} is (λ, κ_i)-saturated over N^a_i. Let $\bar{\delta}^i = (\kappa_i, \lambda, \lambda)$.

We can find a $\leq_{\mathfrak{K}}$-increasing sequence $\langle M^x_i : i \leq \alpha\rangle$ for $x \in \{a,b,c\}$, continuous for $x = a, b$ such that $i < \alpha \Rightarrow M^b_i \leq_{\mathfrak{K}} M^c_i \leq_{\mathfrak{K}} M^b_{i+1}$ and $M^b_\alpha \leq_{\mathfrak{K}} M^c_\alpha$ and $NF_{\lambda,\bar{\delta}^i}(M^a_i, M^a_{i+1}, M^c_i, M^b_{i+1})$ by choosing M^a_i, M^b_i, M^c_i by induction on i. By Case A we know that $NF_\lambda(M^a_0, M^a_\alpha, M^b_0, M^c_\alpha)$ holds.

We can now choose an isomorphism f^a_0 from N^a_0 onto M^a_0 (exists, as K is categorical in λ) and then a $\leq_{\mathfrak{K}}$-embedding of N^b_0 into M^b_0 extending f^a_0. Next we choose, by induction on $i \leq \alpha, f^a_i$ an isomorphism from N^a_i onto M^a_i such that: $j < i \Rightarrow f^a_j \subseteq f^a_i$, possible by "uniqueness of the (λ, κ_i)-saturated model over M^a_i" (see 0.29).

Now we choose, by induction on $i \leq \alpha$, a $\leq_{\mathfrak{K}}$-embedding f_i^b of N_i^b into M_i^b extending f_i^a and f_j^b for $j < i$. For $i = 0$ we have done it, for i limit use $\bigcup_{j<i} f_j^b$, and lastly for i a successor ordinal let $i = j + 1$. Now we have

$(*)_2$ $NF_\lambda(M_i^a, M_{i+1}^a, f_i^b(N_i^b), M_{i+1}^b)$.

[Why? Because $NF_{\lambda,\bar{\delta}^i}(M_i^a, M_{i+1}^a, M_i^c, M_{i+1}^b)$ by the choice of the M_ζ^x's, hence by 8.14 we have $NF_\lambda(M_i^a, M_{i+1}^a, M_i^c, M_{i+1}^b)$ and, as $M_i^a \leq_{\mathfrak{K}} f_i^b(N_i^b) \leq M_i^b, M_i^c$, by 8.15 we get $(*)_2$.]

By $(*)_2$ and the uniqueness of smooth amalgamation, i.e. 8.13, there is f_i^b as required. Hence without loss of generality f_α^b is the identity, so we have $N_0^a = M_0^a, N_\alpha^a = M_\alpha^a, N_0^b \leq_{\mathfrak{K}} M_0^b, N_\alpha^b \leq_{\mathfrak{K}} M_\alpha^b$; also as noted above $NF_\lambda(M_0^a, M_\alpha^a, M_0^b, M_\alpha^b)$ holds, so by monotonicity, i.e. 8.15, we get $NF_\lambda(N_0^a, N_\alpha^a, N_0^b, N_\alpha^b)$ as required.

Case C: General case.

We can find M_i^ℓ for $\ell < 3, i \leq \alpha$ such that:

(a) $M_i^\ell \in K_\lambda$,

(b) for each $\ell < 3, M_i^\ell$ is $\leq_{\mathfrak{K}}$-increasing in i,

(c) $M_i^0 = N_i^a$,

(d) $M_{i+1}^{\ell+1}$ is (λ, λ)-saturated over $M_{i+1}^\ell \cup M_i^{\ell+1}$ for $\ell < 2, i < \alpha$,

(e) $NF_\lambda(M_i^\ell, M_{i+1}^\ell, M_i^{\ell+1}, M_i^{\ell+1})$ for $\ell < 2, i < \alpha$,

(f) $M_0^{\ell+1}$ is (λ, λ)-saturated over M_0^ℓ for $\ell < 2$,

(g) for $\ell < 2$ and $i < \alpha$ limit we have

$$M_i^{\ell+1} \text{ is } (\lambda, \lambda)\text{-saturated over } \bigcup_{j<i} M_j^{\ell+1} \cup M_i^\ell,$$

(h) for $i < \alpha$ limit we have

$$NF_\lambda\Big(\bigcup_{j<i} M_j^1, M_i^1, \bigcup_{j<i} M_j^2, M_i^2\Big).$$

[How? As in the proof of 8.8.]

Now note:

$(*)_4$ $M_i^{\ell+1}$ is $(\lambda, \mathrm{cf}(\upsilon \times (1+i)))$-saturated over M_i^ℓ.

[Why? If $i = 0$ by clause (f), if i is a successor ordinal by clause (d) and if i is a limit ordinal, then by clause (g).]

$(*)_5$ For $i < \alpha, NF_\lambda(M_i^0, M_{i+1}^0, M_i^2, M_{i+1}^2)$.

[Why? We use Case B for $\alpha = 2$ with $M_i^0, M_{i+1}^0, M_i^1, M_{i+1}^1, M_i^2, M_{i+1}^2$ here standing for $N_0^a, N_0^b, N_1^a, N_1^b, N_2^a, N_2^b$ there.]

Now we continue as in Case B (using $f_i^a = \mathrm{id}_{N_i^a}$ and defining by induction on i a $\leq_{\mathfrak{K}}$-embedding f_i^b of N_i^b into M_i^c).

(2) For each i by part (1) the sequences $\langle M_{\beta,i} : \beta \leq \alpha_1 \rangle, \langle M_{\beta,i+1} : \beta \leq \alpha_1 \rangle$ we get $M_{\alpha_1} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup}} M_{0,i+1}$, hence by symmetry (i.e. 8.13) we have $M_{0,i+1} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup}} M_{\alpha_1,i}$. Applying part (1) to the sequences $\langle M_{0,j} : j \leq \alpha_2 \rangle, \langle M_{\alpha_1,j} : j \leq \alpha_2 \rangle$ we get $M_{0,\alpha_2} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup}} M_{\alpha_1,0}$, hence by symmetry (i.e. 8.13) we have $M_{\alpha_1,0} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup}} M_{0,\alpha_2}$; by monotonicity, i.e. 8.15 (or restriction of the matrix), we get the desired conclusion. $\blacksquare_{8.18}$

8.19 CONCLUSION: Assume $\langle N_i^\ell : i \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous for $\ell = 0, 1$ where $N_i^\ell \in K_\lambda$ and N_{i+1}^1 is (λ, κ_ℓ)-saturated over $N_{i+1}^0 \cup N_i^1$ and $\mathrm{NF}_\lambda(N_i^0, N_i^1, N_{i+1}^0, N_{i+1}^1)$.

Then N_α^1 is $(\lambda, \mathrm{cf}(\sum_{i<\alpha} \kappa_i))$-saturated over $N_\alpha^0 \cup N_0^1$ (if α is a limit ordinal, "N_{i+1}^1 is universal over $N_{i+1}^0 \cup N_i^1$" suffices).

Proof: The case α not limit is trivial, so assume α is a limit ordinal. We choose, by induction on $i \leq \alpha$, a sequence $\langle M'_{i,\varepsilon} : \varepsilon \leq \varepsilon(i) \rangle$ such that:

(a) $\langle M_{i,\varepsilon} : \varepsilon \leq \varepsilon(i) \rangle$ is (strictly) $<_{\mathfrak{K}}$-increasing continuous,

(b) $N_i^0 \leq_{\mathfrak{K}} M_{i,\varepsilon} \leq_{\mathfrak{K}} N_i^1$,

(c) $N_i^0 = M_{i,0}$,

(d) $\varepsilon(i)$ is (strictly) increasing continuous in i,

(e) $j < i \;\&\; \varepsilon \leq \varepsilon(j) \Rightarrow M_{i,\varepsilon} \cap N_j^1 = M_{j,\varepsilon}$,

(f) $\varepsilon(0) = 1, M_{i,1} = N_0^1$,

(g) for $i > 0, \lambda$ divides $\varepsilon(i)$,

(h) $N_\ell^i \leq_{\mathfrak{K}} M_{i+1,\varepsilon(i)+1}$.

If we succeed, then $\varepsilon(\alpha)$ is divisible by λ and $\langle M_{i,\varepsilon} : \varepsilon \leq \varepsilon(\alpha) \rangle$ is (strictly) $<_{\mathfrak{K}}$-increasing continuous, $M_{\alpha,0} = N_\alpha^0$, and $M_{\alpha,\varepsilon(\alpha)} \leq_{\mathfrak{K}} N_\alpha^1$, but it includes N_i^1 for $i < \alpha$ hence (as α is a limit ordinal) it includes $\bigcup_{i<\alpha} N_i^1 = N_\alpha^1$; and by 7.6 we conclude that $N_\alpha^1 = M_{\alpha,\varepsilon(\alpha)}$ is $(\lambda, \mathrm{cf}(\alpha))$-saturated over $M_{\alpha,1}$ hence over $N_\alpha^0 \cup N_0^1$ (both $\prec M_{\alpha,i}$).

For $i = 0$ and i limit there is not much to do. For i successor we use 8.20 below.

8.20 CONCLUSION: (1) If $NF_\lambda(N_0, N_1, N_2, N_3)$ and $\langle M_{0,\varepsilon} : \varepsilon \leq \varepsilon(*) \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous, $N_0 \leq_{\mathfrak{K}} M_{0,\varepsilon} \leq_{\mathfrak{K}} M_2$ then we can find $\langle M_{1,\varepsilon} : \varepsilon \leq \varepsilon(*) \rangle$ and N'_3 such that:

(a) $N_3 \leq_{\mathfrak{K}} N'_3 \in K_\lambda$,

(b) $\langle M_{1,\varepsilon} : \varepsilon \leq \varepsilon(*) \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous,

(c) $M_{1,\varepsilon} \cap N_2 = M_{0,\varepsilon}$,

(d) $N_1 \leq_{\mathfrak{K}} M_{1,\varepsilon} \leq_{\mathfrak{K}} N_3'$,

(e) if $M_{0,0} = N_0$ then $M_{1,0} = N$.

(2) If N_3 is universal over $N_1 \cup N_2$, then without loss of generality $N_3 = N_3'$.

Proof: (1) Straightforward by uniqueness.

(2) Follows by (1). $\blacksquare_{8.19}$, $\blacksquare_{8.20}$

9. Nice extensions in K_{λ^+}

9.1 Hypothesis: Assume Hypothesis 8.1.

So by §8 we have reasonable control on smooth amalgamation in K_λ. We use this to define "nice" extensions in K_{λ^+}. This is treated again in §10.

9.2 Definition: (1) Let $M_0 \leq^*_{\lambda^+} M_1$ mean:

(a) $M_\ell \in K_{\lambda^+}$, for $\ell = 0, 1$,

(b) we can find $\bar{M}^\ell = \langle M_i^\ell : i < \lambda^+ \rangle$, a representation of M^ℓ, so $M_i^\ell \in K_\lambda$ (and M_i^ℓ is $\leq_{\mathfrak{K}}$-increasing continuously and $M_\ell = \bigcup_{i<\lambda^+} M_i^\ell$) such that: $NF_\lambda(M_i^0, M_{i+1}^0, M_i^1, M_{i+1}^1)$ for $i < \lambda^+$.

(2) Let $M_0 <^+_{\lambda^+,\kappa} M_1$ mean $M_0 \leq^*_{\lambda^+} M_1$ by some witnesses M_i^ℓ (for $i < \lambda^+, \ell < 2$) such that $NF_{\lambda,\langle\kappa,1,\kappa\rangle}(M_i^0, M_{i+1}^0, M_i^1, M_{i+1}^1)$. If $\kappa = \lambda$, we omit it.

9.3 Claim: (1) *If $M_0 \leq^*_{\lambda^+} M_1$ and $\bar{M}^\ell = \langle M_i^\ell : i < \lambda^+ \rangle$ is a representation of M_ℓ (as in 8.18) then for some club E of λ^+, for every $\alpha < \beta$ from E we have $NF_\lambda(M_\alpha^0, M_\beta^0, M_\alpha^1, M_\beta^1)$.*

(2) *Similarly for $<^+_{\lambda^+,\kappa}$; if $M_0 <^*_{\lambda^+,\kappa_\ell} M_1, \bar{M}^\ell = \langle \bar{M}_i^\ell : i < \lambda^+ \rangle$ a representation of M_ℓ for $\ell = 1, 2$ then for some club E of λ^+ for every $\alpha < \beta$ from E we have $NF_{\lambda,\langle \mathrm{cf}(\alpha),1,\mathrm{cf}(\alpha)\rangle}(M_\alpha^0, M_\beta^0, M_\alpha^1, M_\beta^0)$.*

(3) The κ in Definition 9.2(2) does not matter. In fact, if $\langle M_i^\ell : i < \lambda^+ \rangle$ are as in 9.2(1), then for some club E of λ^+ we have: $\alpha \in E \Rightarrow M_\alpha^1 \cap M_0 = M_\alpha^0$ and $\alpha < \beta$ & $\alpha \in E$ & $\beta \in E \Rightarrow [M_\beta^1$ is $\mathrm{cf}(\beta)$-saturated over $M_\beta^0 \cup M_\alpha^1]$.

Proof: (1) Straightforward by 8.18.

(2) Easy using 9.19.

(3) By 8.19. (We could have used 7.8.) $\blacksquare_{9.3}$

9.4 Claim: (1) *For every $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ and $M_0 \in K_{\lambda^+}$ for some $M_1 \in K_{\lambda^+}$ we have $M_0 <^+_{\lambda^+,\kappa} M_1$.*

(2) *$\leq^*_{\lambda^+}$ and $<^+_{\lambda^+,\kappa}$ are transitive and $M_1 <^+_{\lambda^+,\kappa} N \Rightarrow M \leq^*_{\lambda^+,\kappa} N$.*

(3) *If $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ and $M_0 \leq^*_{\lambda^+} M_2$ then $M_0 \leq^*_{\lambda^+} M_1$.*

(4) [transitivity] *If* $M_0 \leq^*_{\lambda^+} M_1 <^+_{\lambda^+,\kappa} M_2$ *then* $M_0 <^+_{\lambda^+,\kappa} M_2$.

Proof: (1) Let $\langle M^0_i : i < \lambda^+ \rangle$ be a representation of M_0 such that M^0_{i+1} is (λ, κ)-saturated over M^0_i. We choose, by induction on $i, M^1_i \in K_\lambda$ such that $\langle M^1_i : i < \lambda^+ \rangle$ is $<_{\mathfrak{K}}$-increasing continuously, $M^0_i \leq_{\mathfrak{K}} M^1_i, M^1_i \cap M_0 = M^0_i$ and $NF_{\lambda,\langle \kappa,1,\kappa \rangle}(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$. We can do it by 7.7(4).

(2) Concerning $<^+_{\lambda^+,\kappa}$ use 9.3 and 8.18 (i.e. transitivity for smooth amalgamations). Now the proof for $\leq^*_{\lambda^+}$ is similar.

(3) By monotonicity for smooth amalgamations, i.e. 8.15.

(4) Routine verification. $\blacksquare_{9.4}$

9.5 Claim: (1) *If* $M_0 \leq^*_{\lambda^+} M_\ell$ *for* $\ell = 1, 2, \kappa = \mathrm{cf}(\kappa) \leq \lambda$ *and* $a \in M_2 \backslash M_0$ *then for some* M_3 *and* f *we have:* $M_1 <^+_{\lambda^+,\kappa} M_3$ *and* f *is an* $\leq_{\mathfrak{K}}$*-embedding of* M_2 *into* M_3 *over* M_0 *with* $f(a) \notin M_1$*, moreover,* $f(M_2) \leq^*_{\lambda^+} M_3$.

(2) [uniqueness] *Assume* $M_0 <^+_{\lambda^+,\kappa} M_\ell$ *for* $\ell = 1, 2$; *then there is an isomorphism* f *from* M_1 *onto* M_2 *over* M_0.

Proof: We first prove part (2).

(2) By 9.3(1) + (2) there are representations $\bar{M}^\ell = \langle M^\ell_i : i < \lambda^+ \rangle$ of M_ℓ for $\ell < 3$ such that: $M^\ell_i \cap M_0 = M^\ell_0$ and $NF_{\lambda,\langle \kappa,\mathrm{cf}(\kappa\times i),\kappa \rangle}(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$.

Now we choose, by induction on $i < \lambda$, an isomorphism f_i from M^1_i onto M^2_i, increasing with i and being the identity over M^0_i. For $i = 0$ use "M^ℓ_0 is (λ, κ)-saturated over M^0_0 for $\ell = 1, 2$" which holds by 7.1. For i limit take unions, for i successor ordinal use uniqueness Claim 8.9.

Proof of part (1): Let $\kappa = \aleph_0$, by 9.4(1) there are for $\ell = 1, 2$ models $N^*_\ell \in K_{\lambda^+}$ such that $M_\ell <^+_{\lambda^+,\kappa} N^*_\ell$. Now let $\bar{M}^\ell = \langle M^\ell_i : i < \lambda^+ \rangle$ be a representation of M_ℓ for $\ell = 0, 1, 2$ and let $\bar{N}^\ell = \langle N^\ell_i : i < \lambda^+ \rangle$ be a representation of N^*_ℓ for $\ell = 1, 2$. By 9.4(4) and 9.3(3), without loss of generality $NF_\lambda(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$ for $\ell = 1, 2$ and $NF_{\lambda,\langle \kappa,1,\kappa \rangle}(M^\ell_i, M^\ell_{i+1}, N^\ell_i, N^\ell_{i+1})$, respectively. Now clearly N^ℓ_0 is (λ, κ)-saturated over M^ℓ_0, hence over M^0_0 (for $\ell = 1, 2$), so there is an isomorphism f_0 from N^2_0 onto N^1_0 extending $\mathrm{id}_{M^0_0}$ and $f(a) \notin M^1_0$.

We continue as in the proof of part (2). In the end $f = \bigcup_{i<\lambda^+} f_i$ is an isomorphism of N_2 onto N_1 over M_0 and as $f^1_0(a)$ is well defined and in $N^1_0 \backslash M^1_0$, clearly $f(a) = f_0(a) \notin M_1$, as required. $\blacksquare_{9.5}$

9.6 Claim: *If* δ *is a limit ordinal* $< \lambda^{+2}$ *and* $\langle M_i : i < \delta \rangle$ *is a* $\leq^*_{\lambda^+}$*-increasing continuous* *then* $M_i \leq^*_{\lambda^+} \bigcup_{j<\delta} M_j$ *for each* $i < \delta$.

Proof: We prove it by induction on δ. Now if C is a club of δ with $i \in C$, then we can replace $\langle M_j : j < \delta \rangle$ by $\langle M_j : j \in C \rangle$ so, without loss of generality,

$\delta = \mathrm{cf}(\delta)$, hence $\delta \leq \lambda^+$; clearly it is enough to prove $M_0 \leq^*_{\lambda^+} \bigcup_{j<\delta} M_j$. Let $\langle M^i_\zeta : \zeta < \lambda^+\rangle$ be a representation of M_i.

Case A: $\delta < \lambda^+$.

Without loss of generality (see 9.3(1)) for every $i < j < \delta$ and $\zeta < \lambda^+$ we have: $M^j_\zeta \cap M_i = M^i_\zeta$ and $NF_\lambda(M^i_\zeta, M^i_{\zeta+1}, M^j_\zeta, M^j_{\zeta+1})$. Let $M^\delta_\zeta = \bigcup_{i<\delta} M^i_\zeta$, so $\langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ is a $\leq_{\mathfrak{K}}$-increasingly continuous sequence of members of K_λ with limit M_δ, and for $i < \delta, M^\delta_\zeta \cap M_i = M^i_\zeta$. By symmetry (see 8.16) we have $NF_\lambda(M^i_\zeta, M^{i+1}_\zeta, M^i_{\zeta+1}, M^{i+1}_{\zeta+1})$, so as $\langle M^i_\zeta : i \leq \delta\rangle$, $\langle M^i_{\zeta+1} : i \leq \delta\rangle$ are $\leq_{\mathfrak{K}}$-increasingly continuous by 8.18 we know $NF_\lambda(M^0_\zeta, M^\delta_\zeta, M^0_{\zeta+1}, M^\delta_{\zeta+1})$, hence by symmetry (8.16) we have $NF_\lambda(M^0_\zeta, M^0_{\zeta+1}, M^\delta_\zeta, M^\delta_{\zeta+1})$.

So $\langle M^0_\zeta : \zeta < \lambda^+\rangle, \langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ are witnesses to $M_0 \leq^*_{\lambda^+} M_\delta$.

Case B: $\delta = \lambda^+$.

By 9.3(1) (using normality of the club filter, restricting to a club of λ^+ and renaming), without loss of generality for $i < j \leq 1 + \zeta < 1 + \xi < \lambda^+$ we have $M^j_\zeta \cap M_i = M^i_\zeta$, and $NF_\lambda(M^i_\zeta, M^i_\xi, M^j_\zeta, M^j_\xi)$. Let us define $M^{\lambda^+}_i = \bigcup_{j<1+i} M^j_i$. So $\langle M^{\lambda^+}_i : i < \lambda^+\rangle$ is a representation of $M^\lambda_{\lambda^+} = M_\delta$ and continue as before. $\blacksquare_{9.6}$

9.7 Claim: *Assume $M_0 <^+_{\lambda^+,\kappa} M_2$ and $a \in M_2 \backslash M_0$, and for some $N \leq_{\mathfrak{K}} M_0$ we have: $N \in K_\lambda$ and $tp(a, N, M_2)$ is minimal. Then we can find M_1, $\bar{M}^0 = \langle M_{0,i} : i < \lambda^+\rangle, \bar{M}^1 = \langle M_{1,i} : i < \lambda^+\rangle$ such that:*

(a) *$\bar{M}^0$ is a $\leq_{\mathfrak{K}}$-representation of M_0,*
(b) *$\bar{M}^1$ is a representation of $M_1 (\in K_{\lambda^+}), a \in M_{1,i}$, for all i,*
(c) *$M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$,*
(d) *for $i < \lambda^+$ we have $NF_{\lambda,\langle\lambda,1,1\rangle}(M_{0,i}, M_{0,i+1}, M_{1,i}, M_{1,i+1})$ (hence $M_{\ell,i} = M_{\ell+1,i} \cap M_\ell$),*
(e) *$(M_{0,i}, M_{1,i}, a) \in K^3_\lambda$ is minimal and reduced.*

Proof: Let $\langle M_{0,i} : i < \lambda^+\rangle, \langle M_{2,i} : i < \lambda^+\rangle$ be representations of M_0, M_2 respectively, as required in 9.2(2), and without loss of generality $N \leq_{\mathfrak{K}} M_{0,0}$ and $a \in M_{2,0}$. We now choose, by induction on $\zeta < \lambda^+$, an ordinal $i(\zeta)$ and models $M_{1,i(\zeta)}, M_{3,i(\zeta)}$ such that:

(A) $i(\zeta) < \lambda^+$ is increasing continuous in ζ and $a \in M_{2,i(0)} \backslash M_{0,i(0)}, N \leq_{\mathfrak{K}} M_{0,i(0)}$,
(B) $M_{0,i(\zeta)} \leq_{\mathfrak{K}} M_{1,i(\zeta)} \leq_{\mathfrak{K}} M_{3,i(\zeta)}$ and $M_{2,i(\zeta)} \leq_{\mathfrak{K}} M_{3,i(\zeta)}$,
(C) $a \in M_{1,i(0)}$ and $(M_{0,i(\zeta)}, M_{1,i(\zeta)}, a)$ is minimal and reduced,
(D) for $\xi < \zeta$ and $(\ell, m) \in \{(0,1),(0,2),(1,3),(2,3)\}$ we have $NF_\lambda(M_{\ell,i(\xi)}, M_{\ell,i(\zeta)}, M_{m,i(\xi)}, M_{m,i(\zeta)})$,
(E) $M_{1,i(\zeta)}, M_{3,i(\zeta)}$ is $\leq_{\mathfrak{K}}$-increasing continuous in ζ.

For $\zeta = 0$ note that, for $i(0) < \lambda^+, a \in M_{2,i(0)}$ and $M_{2,i(0)}$ is universal over $M_{0,i(0)}$.

For ζ limit let $i(\zeta) = \bigcup_{\xi<\zeta} i(\xi)$ and $M_{1,i(\zeta)} = \bigcup_{\xi<\zeta} M_{1,i(\zeta)}$.

For $\zeta = \xi + 1$, there is $i(\zeta) \in (i(\xi), \lambda^+)$ and a model N_ζ such that $M_{1,i(\xi)} \leq_{\mathfrak{K}} N_\zeta \in K_\lambda$ and $\leq_{\mathfrak{K}}$-embedding f of $M_{0,i(\zeta)}$ into $N_\zeta, f \restriction M_{0,i(\zeta)}$ the identity and $(f(M_{0,i(\xi)}), N_\zeta, a)$ is minimal and reduced. By uniqueness (i.e. Claim 8.2) we can find such N satisfying $(\exists M)(N \leq_{\mathfrak{K}} M \in K_\lambda \;\&\; M_{1,i(\zeta)} \leq_{\mathfrak{K}} M)$. So we can carry the induction.

Lastly, by uniqueness of $<^+_{\lambda,\kappa}$ we can make $M_3 = \bigcup_{\zeta<\lambda^+} M_{3,i(\zeta)}$ to be $\leq_{\mathfrak{K}} M_2$ as required. $\blacksquare_{9.7}$

9.8 Definition: If (M_0, M_1, a) are as in 9.7(a)–(e) we say (M_0, M_1, a) is λ^+-**locally reduced nice** and minimal (λ^+-**l.r.n.m.**). We omit "nice" if we omit clause (d).

9.9 CLAIM: *If (M_0, M_1, a) is λ^+-l.r.n.m. <u>then</u> $(M_0, M_1, a) \in K^3_{\lambda^+}$ is reduced.*

Proof: Check.

We can also have

9.10 CLAIM: $M_0 <^+_{\lambda^+,\kappa} M_1$ <u>if and only if</u> *we can find $\langle M^*_j, a_j : j < \lambda^+ \times \kappa\rangle$ such that:*

(a) *M^*_j is $\leq_{\mathfrak{K}}$-increasing continuous (in K_{λ^+}),*
(b) *(M^*_j, M^*_{j+1}, a_j) is λ^+-l.r.n.m.,*
(c) *$M^*_0 = M_0$, $M^*_{\lambda^+\times\kappa} = M_1$,*
(d) *for some $N \leq M_0, N \in K_\lambda$ and minimal reduced $p \in S(N)$, for every j, a_j realizes p.*

Proof: We can find $\langle M^*_j : j \leq \lambda^+ \times \kappa\rangle$ satisfying clauses (a), (b) and (d). Clearly if $\langle N^\ell_\alpha : \alpha < \lambda^+\rangle$ is $<_{\mathfrak{K}}$-increasing continuous in K_λ, $N^0_\alpha \leq_\kappa N^1_\alpha, p = \mathrm{tp}(a, N^0_0, N^1_0)$ is minimal then for club of α, $\mathrm{tp}(a, N^0_\alpha, N^1_\alpha)$ in a minimal reduced extension of p. Hence 5.6, easily $M^*_0 <^+_\lambda M^*_{\lambda^+}$. Now by the uniqueness ($=$ 9.5(2)) + categoricity of K in λ^+, we are done. $\blacksquare_{9.10}$

9.11 CLAIM: *In $(K_{\lambda^+}, <^*_{\lambda^+})$ we have disjoint amalgamation.*

Proof: First redo 9.5 assuming (M_0, M_ℓ, a_ℓ) for $\ell = 1, 2$ is λ^+-l.r.n.m., and getting $a_1 \notin f(M_2), f(a_2) \notin M_1$ (just embed both into some M^*, $M <^+_{\lambda^+,\kappa} M^*$; and we can start with this). By 9.9 we get $M_1 \cap f(M_2) = M_0$, so we have disjoint amalgamation. By 9.10 and chasing arrows we get it in general. $\blacksquare_{9.11}$

Remark: This is like the proof of disjoint amalgamations in 5.11.

10. Non-structure for $\leq^*_{\lambda^+}$

10.1 HYPOTHESIS: Assume hypothesis 8.1 and the further model theoretic properties deduced since then (including 6.7 but not 6.12).

It would have been nice to prove all disjoint amalgamations in K_λ are NF_λ, but this is, at this point, not clear. But as we look upward (i.e. we want to prove the statement on $K_{>\lambda^+}$) and $<^*_{\lambda^+}$ is very nice, it will be essentially just as well if for $M, N \in K_{\lambda^+}$ we have $M \leq_{\mathfrak{K}} N \Rightarrow M \leq^*_{\lambda^+} N$. Our intention is to assume $M^* \leq_{\mathfrak{K}} N^*$ is a counterexample of this statement and we would like to say that in a sense this implies the existence of many types over M^* so that we can construct many models in λ^{+2}. Note: Building models in $K_{\lambda^+}, K_{\lambda^{++}}$ by approximations in K_λ is nice if we use the smooth amalgamation but we do not have it for non-smooth ones. So we shall use $M^* \in K_{\lambda^+}$ being saturated so it has many automorphisms.

10.2 CLAIM: (1) *Assume $M_1 \leq_{\mathfrak{K}} M_2$ are in K_{λ^+}. Then we can find $M_0 \in K_{\lambda^+}$ such that $M_0 <^+_{\lambda,\kappa} M_1$ and $M_0 \leq^*_{\lambda^+} M_2$.*
(2) *Also we can find $\langle M_{0,i} : i < \lambda^+\rangle$, an $\leq_{\mathfrak{K}}$-increasingly continuous sequence of members of K_{λ^+} such that $M_{0,i} <^*_{\lambda^+} M_{0,i+1}$ and $\bigcup_{i<\lambda^+} M_{0,i} = M_1$ and $i < \lambda^+ \Rightarrow M_{0,i} \leq^*_{\lambda^+} M_2$.*

Proof: Let $(M^*, N^*) \in K^{3,\mathrm{uq}}_\lambda$ be from 8.1(2). Let $\langle M_{\ell,i} : i < \lambda^+\rangle$ be a representation of M_ℓ for $\ell = 1, 2$ and, without loss of generality, $M_{\ell,i+1}$ is (λ, λ)-saturated over $M_{\ell,i}$ for $\ell = 1, 2$ and $M_{2,i} \cap M_1 = M_{1,i}$.
(1) Now choose, by induction on $i, M_{0,i}$ such that:

- (a) $M_{0,i} \leq_{\mathfrak{K}} M_{1,i}$,
- (b) $M_{0,i}$ is $\leq_{\mathfrak{K}}$-increasing continuous,
- (c) $M_{0,i+1} \cap M_{1,i} = M_{0,i}$,
- (d) $M_{1,i+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-saturated over $M_{1,i} \cup M_{0,i+1}$,
- (e) $(M_{0,i}, M_{0,i+1}) \cong (M^*, N^*)$.

There is no problem to carry the definition. Now let $M_0 = \bigcup_{i<\lambda^+} M_{0,i}$, so $M_0 <^+_{\lambda^+} M_1$ and $M_0 \leq^*_{\lambda^+} M_2$ are checked by their definitions noting clause (e), the choice of (M^*, N^*) and the definition of NF_λ in 8.2.
(2) We choose, by induction on $i < \lambda^+, \langle M^*_{\varepsilon,i} : \varepsilon \leq 1 + i\rangle$ such that:

- (a) $M^*_{1+i,i} = M_{1,\lambda\times(1+\varepsilon)\times i}$,
- (b) for each ε the sequence $\langle M^*_{\varepsilon,j} : \varepsilon \leq j \leq i\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous,
- (c) for each i the sequence $\langle M^*_{\varepsilon,i} : \varepsilon \leq 1 + i\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous,
- (d) $M^*_{\varepsilon,i} \cap M^*_{\zeta,j} = M^*_{\min\{\varepsilon,\zeta\},\min\{i,j\}}$,
- (e) $M^*_{\varepsilon+1,i+1}$ is $(\lambda, \mathrm{cf}(\lambda \times (1 + \varepsilon)))$-saturated over $M^*_{\varepsilon,i+1} \cup M^*_{\varepsilon+1,i}$,

(f) $NF_{\lambda,\langle\lambda,1,\lambda\rangle}(M^*_{\varepsilon,i}, M^*_{\varepsilon+1,i}, M^*_{\varepsilon,i+1}, M^*_{\varepsilon+1,i+1})$,

(g) for $\varepsilon < 1+i$ we have $NF_\lambda(M^*_{\varepsilon,i}, M^*_{\varepsilon,i+1}, M_{2,\lambda\times\lambda\times i}, M_{2,\lambda\times\lambda\times(i+1)})$.

For $i = 0, i$ limit, there is no problem; for $i = j+1$ first choose $N_{i,\zeta} \leq_{\mathfrak{K}} M_{1,\lambda\times\lambda\times i+\lambda\times\zeta}$ for $\zeta \leq \upsilon$, $\leq_{\mathfrak{K}}$-increasing continuous, $N_{i,\zeta} \in K_\lambda, N_{i,0} = M_{1,j}$ $(= M^*_{1+i,i})$, $(N_{i,\zeta}, N_{i,\zeta+1}) \cong (M^*, N^*)$ and $N_{i,\zeta+1} \cap M_{2,\lambda\times\lambda\times i+\lambda\times\zeta} = N_{i,\varepsilon,\zeta}$.

Now by 7.9, without loss of generality $N_{i,\lambda}$ is (λ,λ)-saturated over $M_{1,j}$, and we choose it as $M^*_{1+j,i}$, and we choose $M^*_{1,i+1}$ as $M_{1+i,i}$; note that clauses (a) and (f) hold. Now we can find $M_{\varepsilon,i}$ for $\varepsilon < 1+j$ as in 8.8 and use uniqueness of the $(\lambda, \lambda\times(1+i))$-saturated model over $M_{1,j}$. $\blacksquare_{10.2}$

10.3 Conclusion: Assume $M \leq_{\mathfrak{K}} N$ are from K_{λ^+}. If $\langle M_i : i < \lambda^+\rangle$ is $\leq^*_{\lambda^+}$-increasing continuous and, for each i for some N_i, we have $M_i <^+_{\lambda^+} N_i \leq^*_{\lambda^+} M_{i+1}$ <u>then</u> for some (M', N') we have:

$$(M', N') \cong (M, N),$$
$$M' = \bigcup_{i<\lambda} M_i,$$
$$i < \lambda^+ \Rightarrow M_i \leq^*_{\lambda^+} N'.$$

Proof: By 10.2(2) and by the implication $M^a \leq^*_{\lambda^+} M^b \leq^+_{\lambda^+} M^c \Rightarrow M^a <^+_{\lambda^+} M^c$ and by the uniqueness of M'' over M' when $M' <^+_{\lambda^+} M''$. $\blacksquare_{10.3}$

10.4 Lemma: *If* $\leq_{\mathfrak{K}}\restriction K_{\lambda^+}$ *is not* $\leq^*_{\lambda^+}$ <u>then</u> $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.

Proof: Let $S \subseteq \{\delta < \lambda^{+2} : \mathrm{cf}(\delta) = \lambda^+\}$ be stationary. We shall construct below a model $M_S \in K_{\lambda^{+2}}$ such that, from the isomorphism type of M^S_1, we can reconstruct $S/\mathcal{D}_{\lambda^{+2}}$; this clearly suffices. Choose $M \leq_{\mathfrak{K}} N$ from K_{λ^+} such that $\neg(M \leq^*_{\lambda^+} N)$, so by 9.4(3), without loss of generality, $|N \backslash M| = \lambda^+$.

We choose by induction on $\alpha < \lambda^{+2}$ a model M^S_α such that:

(a) $M^S_\alpha \in K_{\lambda^+}$ has universe $\lambda \times (1+\alpha)$,

(b) for $\beta < \alpha$ we have $M^S_\beta \leq_{\mathfrak{K}} M^S_\alpha$,

(c) if $\alpha = \beta+1$, $\beta \notin S$ then $M^S_\beta <^+_{\lambda^+} M^S_\alpha$,

(d) if $\alpha = \beta+1, \beta \in S$ then $(M^S_\beta, M^S_\alpha) \cong (M, N)$,

(e) if $\beta < \alpha, \beta \notin S$ then $M^S_\beta \leq^*_{\lambda^+} M^S_\alpha$.

We use freely the transitivity (9.4(4)) and continuity (9.6) of $\leq^*_\lambda$ and $[M^a \leq_{\mathfrak{K}} M^b \leq_{\mathfrak{K}} M^c$ in $K_\lambda, \neg(M^a \leq^*_{\lambda^+} M^b \Rightarrow \neg(M^a \leq^*_{\lambda^+} M^c)]$ (9.4(3)).

The cases $\alpha = 0, \alpha$ is a limit ordinal and $\alpha = \beta+1, \beta \notin S$ present no problem.

For $\alpha = \beta+1, \beta \in S$, so $\mathrm{cf}(\beta) = \lambda^+$. Let $\langle \gamma_i : i < \lambda^+\rangle$ be increasing continuous with limit β and $\mathrm{cf}(\gamma_i) \leq \lambda^+$, hence $\gamma_i \notin S$. By 8.2(2), without loss of generality, $M_{\gamma_i} <^*_{\lambda^+} M_{\gamma_{i+1}}$. Now use 10.3 (and the uniqueness (9.5(2))). $\blacksquare_{10.4}$

10.5 CONCLUSION: Assume $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ (in addition to hypothesis 10.1). Then

(1) $\leq^*_{\lambda^+} = \leq_{\mathfrak{K}} \restriction K_{\lambda^+}$,

(2) $(K_{\lambda^+}, \leq_{\mathfrak{K}})$ has disjoint amalgamation, so no $M \in K_{\lambda^{+2}}$ is $\leq_{\mathfrak{K}}$-maximal,

(3) $K_{\lambda^{+3}} \neq \emptyset$.

Proof: (1) By 10.4.

(2) By 9.11 (and part (1)).

(3) By 10.5(2) and 2.6(8), with λ there replaced by λ^+ here. $\blacksquare_{10.5}$

So we have finally proved the main theorem. Though not directly contributing to our main theme, we remark on some more consequences of $\leq_{\mathfrak{K}} \restriction K_{\lambda^+} \neq \leq^*_{\lambda^+}$.

10.6 CLAIM: $(*)_0 \Leftrightarrow (*)_1$ *where*

$(*)_0$ *for some* $M \leq_{\mathfrak{K}} N$ *from* K_{λ^+}, *we do not have* $M \leq^*_{\lambda^+} N$,

$(*)_1$ *for some* $M \leq_{\mathfrak{K}} N$ *from* K_{λ^+} *we have:*

if $a \in N \setminus M$ *then* $tp(a, M, N)$ *is not realized in any* M *such that* $M^* \leq^*_{\lambda^+} M \in K_{\lambda^+}$.

10.7 Definition: Assume $M_0 <_{\mathfrak{K}} M_1$ are in K_{λ^+}, and $\bar{M}^\ell = \langle M_{\ell,i} : i < \lambda^+ \rangle$ is a $\leq_{\mathfrak{K}}$-representation of M_ℓ for $\ell = 0, 1$. Let

(a) $\mathcal{S}_0(\bar{M}^0, \bar{M}^1) =$
$$\{\delta < \lambda^+ : M_{1,\delta} \cap M_0 = M_{0,\delta} \text{ and not } NF_\lambda(M_{0,\delta}, M_{0,\delta+1}, M_{1,\delta}, M_1)\},$$

(b) $\mathcal{S}_1(M_0, M_1) = \mathcal{S}_0(\bar{M}^0, \bar{M}^1)/D_{\lambda^+}$ (well defined),

(c) J is the normal ideal on λ^+ generated by sets of the form $\mathcal{S}_0(M^0, M^1)$, where M^0, M^1 are as above.

10.8 COMMENT: An earlier try for 10.4 was:

(1) For every $S \in J$ for some $\bar{M}^0, M^1$ as in 10.7 we have $S = S_0(\bar{M}^0, \bar{M}^1)$.

(2) If $S_1 = S_1(M^0, M^1)$ is stationary, then for some $\bar{M} = \langle M_i : i < \lambda^+ \rangle$, a representation of $M = \bigcup_{i<\lambda^+} M_i \in K_{\lambda^+}$ for every $S \subseteq S_1$ for some $\bar{M}'$, we have $\bar{M}^1 = \langle M^1_i : i < \lambda^+ \rangle, M_1 <_{\mathfrak{K}} M^1 = \bigcup_{i<\lambda^+} M^1_i, M^1_i \cap M = M_i$ and $S_0(\bar{M}, \bar{M}^1) = S \bmod \mathcal{D}_{\lambda^+}$.

(3) If $\leq^*_{\lambda^+} \neq \leq_{\mathfrak{K}} \restriction K_{\lambda^+}$ and $I(\lambda^{+2}, K) < 2^{\lambda^{+2}}$ <u>then</u> for some stationary $S \subseteq \lambda^+$ we have:

(a) $\mathcal{D}_{\lambda^+} \restriction S$ is λ^{++}-saturated,

(b) $\bar{M}^0, \bar{M}^1$ as in 10.3 implies $S_1(\bar{M}^0, \bar{M}^1) \subseteq S \bmod D_{\lambda^+}$.

(4) If $\leq^*_{\lambda^+} \neq \leq_{\mathfrak{K}} \restriction K_{\lambda^+}$ and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}}$ then $I(\lambda^{+2}, K) = 2^{\lambda^{+2}}$.

Proof: (1) First we prove only "$S \subseteq S_0(\bar{M}^0, \bar{M}^1)$". Easy, as $\mathfrak{K}_{\lambda^+}$ has amalgamation and

⊗ if $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ are in K_{λ^+}, $\bar{M}^\ell$ representing the $S_0(\bar{M}^0, \bar{M}^1) \subseteq S_0(\bar{M}^0, \bar{M}^2)$.

Now for equality use part (2).

(2) Similar to the proof of 10.2.

(3) Suppose $S^* = S_1(\bar{M}^0, \bar{M}^2)$ is stationary; let $\bar{S} = \langle S'_\alpha : \alpha < \lambda^{++} \rangle$ be such that $S'_\alpha \in J$. We can build a model $M^{\bar{S}} \in K_{\lambda^{+2}}$ and a representation $\langle M^{\bar{S}}_\alpha : \alpha < \lambda^{++} \rangle$ such that

$$S_1(M_\alpha, M_{\alpha+1}) = S'_\alpha / D_{\lambda^{++}}.$$

(4) By part (3) (using the proof of part (2)).

References

[DvSh 65] K. J. Devlin and S. Shelah, *A weak version of ◇ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel Journal of Mathematics **29** (1978), 239–247.

[Ha 61] A. Hajnal, *Proof of a conjecture of S. Ruziewicz*, Fundamenta Mathematicae **50** (1961), 123–128.

[HaSh 323] B. Hart and S. Shelah, *Categoricity over P for first order T or categoricity for $\phi \in L_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$*, Israel Journal of Mathematics **70** (1990), 219–235.

[KlSh 362] O. Kolman and S. Shelah, *Categoricity of theories in $L_{k,\omega}$, when κ is a measurable cardinal. Part 1*, Fundamenta Mathematicae **151** (1996), 209–240.

[MaSh 285] M. Makkai and S. Shelah, *Categoricity of theories in $L_{\kappa\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[Mw 85] J. A. Makowsky, *Compactness, embeddings and definability*, in *Model-Theoretic Logics* (J. Barwise and S. Feferman, eds.), Springer-Verlag, Berlin, 1985, pp. 645–716.

[MkSh 313] A. H. Mekler and S. Shelah, *Diamond and λ-systems*, Fundamenta Mathematicae **131** (1988), 45–51.

[Sh 600] S. Shelah, *Categoricity in abstract elementary classes: going up inductive step*, a preprint.

[Sh 394] S. Shelah, *Categoricity of abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), 261–294.

[Sh 472] S. Shelah, *Categoricity of theories in $L_{\kappa^*\omega}$, when κ^* is a measurable cardinal. Part II*, Fundamenta Mathematicae, to appear.

[Sh 587] S. Shelah, *Not collapsing cardinals $\leq \kappa$ in $(< \kappa)$-support iterations*, Israel Journal of Mathematics, to appear.

[Sh 87] S. Shelah, See [S 87a] and [Sh 87b].

[Sh 87a] S. Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), 212–240.

[Sh 87b] S. Shelah, *Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), 241–273.

[Sh 460] S. Shelah, *The Generalized Continuum Hypothesis revisited*, Israel Journal of Mathematics **116** (2000), 285–322.

[Sh h] S. Shelah, *Universal classes*, in preparation.

[Sh 133] S. Shelah, *On the number of nonisomorphic models in $L_{\infty,\kappa}$ when κ is weakly compact*, Notre Dame Journal of Formal Logic **23** (1982), 21–26.

[Sh b] S. Shelah, *Proper forcing*, Lecture Notes in Mathematics **940**, Springer-Verlag, Berlin–New York, 1982, xxix+496pp.

[Sh 208] S. Shelah, *More on the weak diamond*, Annals of Pure and Applied Logic **28** (1985), 315–318.

[Sh 88] S. Shelah, *Classification of nonelementary classes. II. Abstract elementary classes*, in *Classification Theory*, Lecture Notes in Mathematics **1292**, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985 (J. T. Baldwin, ed.), Springer, Berlin, 1987, pp. 419–497.

[Sh 300] S. Shelah, *Universal classes*, in *Classification Theory*, Lecture Notes in Mathematics **1292**, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985 (J. T. Baldwin, ed.), Springer, Berlin, 1987, pp. 264–418.

[Sh 311] S. Shelah, *A more general iterable condition ensures $\aleph_1$ is not collapsed*, in preparation.

[Sh c] S. Shelah, *Classification theory and the number of nonisomorphic models*, in *Studies in Logic and the Foundations of Mathematics*, Vol. 92, North-Holland, Amsterdam, 1990, xxxiv+705 pp.

[Sh 351] S. Shelah, *Reflecting stationary sets and successors of singular cardinals*, Archive for Mathematical Logic **31** (1991), 25–53.

[Sh 430] S. Shelah, *Further cardinal arithmetic*, Israel Journal of Mathematics **95** (1996), 61–114.

[Sh f] S. Shelah, *Proper and improper forcing*, in *Perspectives in Mathematical Logic*, Springer, New York, 1998.

[Sh 603] S. Shelah, *Few non-minimal types and non-structure*, Proceedings of the 11-th International Congress of Logic, Methology and Philosophy of Science in Cracow 1999 (Peter Gardenfors, Katargyna Kijania-Placek and Jan Wolenski, eds.), Kluwer Academic Publishers.

[Sh F368] S. Shelah, *More on 600*, in preparation.

[Sh 638] S. Shelah, *More on the weak diamond*, East-West Journal of Mathematics, in press.